440-2 Geometry/Topology: Differentiable Manifolds
Northwestern University
Homework 1
Due January 12 in class.

Throughout this problem set, \( V \) will denote a finite-dimensional vector space over \( \mathbb{R} \).

1) Show that \( T^2(V) = \text{Sym}^2 V \oplus \Lambda^2 V \).

2) Let \((e_1, e_2, e_3)\) be the standard basis of \( \mathbb{R}^3 \). Show that the contravariant 3-tensor \( e_1 \otimes e_2 \otimes e_3 \in T^3(\mathbb{R}^3) \) does not lie in \( \text{Sym}^3 \mathbb{R}^3 \oplus \Lambda^3 \mathbb{R}^3 \).

3) Given \( \alpha \in \Lambda^p \mathbb{R}^3, \beta \in \Lambda^q \mathbb{R}^3, \gamma \in \Lambda^r \mathbb{R}^3 \), show that
\[
(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma).
\]

4) Let \((e_1, e_2, e_3)\) be the standard basis of \( \mathbb{R}^3 \). Consider the isomorphism \( P : \Lambda^2 \mathbb{R}^3 \to \mathbb{R}^3 \) defined by
\[
e_2 \wedge e_3 \mapsto e_1, \quad e_3 \wedge e_1 \mapsto e_2, \quad e_1 \wedge e_2 \mapsto e_3.
\]
Show that given any two vectors \( v, w \in \mathbb{R}^3 = \Lambda^1 \mathbb{R}^3 \), \( P(v \wedge w) \) equals the cross product \( v \times w \).