1) Show that \( T^2(V) = \text{Sym}^2 V \oplus \Lambda^2 V \).

**Solution.** Let \( n = \dim V \). We have that \( \text{Sym}^2 V \) and \( \Lambda^2 V \) are both subspaces of \( T^2(V) \), of dimensions \( \binom{n+1}{2} \) and \( \binom{n}{2} \) respectively. Since these numbers sum to \( n^2 = \dim T^2(V) \), it will be enough to show that \( \text{Sym}^2 V \cap \Lambda^2 V = \{0\} \). The skew-symmetrization and symmetrization maps \( A \) and \( B \) from \( T^2(V) \) to itself that we defined in class clearly satisfy \( A \circ B = 0 \). Therefore if \( v \in \text{Sym}^2 V \cap \Lambda^2 V \) we have

\[
v = Bv = Av = A(Bv) = 0,
\]
as required.

2) Let \( (e_1, e_2, e_3) \) be the standard basis of \( \mathbb{R}^3 \). Show that the contravariant 3-tensor \( e_1 \otimes e_2 \otimes e_3 \in T^3(\mathbb{R}^3) \) does not lie in \( \text{Sym}^3 \mathbb{R}^3 \oplus \Lambda^3 \mathbb{R}^3 \).

**Solution.** Otherwise, we would have

\[
e_1 \otimes e_2 \otimes e_3 = A(e_1 \otimes e_2 \otimes e_3) + B(e_1 \otimes e_2 \otimes e_3).
\]

By definition we have

\[
A(e_1 \otimes e_2 \otimes e_3) = \frac{1}{6} (e_1 \otimes e_2 \otimes e_3 - e_2 \otimes e_1 \otimes e_3 - e_3 \otimes e_2 \otimes e_1 \\
- e_1 \otimes e_3 \otimes e_2 + e_2 \otimes e_3 \otimes e_1 + e_3 \otimes e_1 \otimes e_2),
\]

\[
B(e_1 \otimes e_2 \otimes e_3) = \frac{1}{6} (e_1 \otimes e_2 \otimes e_3 + e_2 \otimes e_1 \otimes e_3 + e_3 \otimes e_2 \otimes e_1 \\
+ e_1 \otimes e_3 \otimes e_2 + e_2 \otimes e_3 \otimes e_1 + e_3 \otimes e_1 \otimes e_2),
\]

and so

\[
A(e_1 \otimes e_2 \otimes e_3) + B(e_1 \otimes e_2 \otimes e_3) = \frac{1}{3} (e_1 \otimes e_2 \otimes e_3 + e_2 \otimes e_3 \otimes e_1 + e_3 \otimes e_1 \otimes e_2),
\]

but it is trivial to see that this is not equal to \( e_1 \otimes e_2 \otimes e_3 \), for example by evaluating both on \( (e^1, e^2, e^3) \).
3) Given \( \alpha \in \Lambda^p V, \beta \in \Lambda^q V, \gamma \in \Lambda^r V \), show that
\[
(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma).
\]

**Solution.** Given any \( v^1, \ldots, v^{p+q+r} \in V^* \), we have
\[
(\alpha \wedge \beta) \wedge \gamma(v^1, \ldots, v^{p+q+r})
= \frac{(p + q + r)!}{(p + q)!r!} \mathcal{A}((\alpha \wedge \beta) \otimes \gamma)(v^1, \ldots, v^{p+q+r})
= \frac{1}{(p + q)!r!} \sum_{\tau \in S_{p+q+r}} \text{sgn}(\tau)(\alpha \wedge \beta) \otimes \gamma(v^{\tau(1)}, \ldots, v^{\tau(p+q+r)})
= \frac{1}{(p + q)!r!} \sum_{\tau \in S_{p+q+r}} \text{sgn}(\tau)(\alpha \wedge \beta)(v^{\tau(1)}, \ldots, v^{\tau(p+q)})\gamma(v^{\tau(p+q+1)}, \ldots, v^{\tau(p+q+r)})
= \frac{1}{(p + q)!p!q!r!} \sum_{\tau \in S_{p+q+r}} \sum_{\sigma \in S_{p+q}} \text{sgn}(\tau)\text{sgn}(\sigma)\alpha(v^{\sigma(1)}, \ldots, v^{\sigma(p)})\beta(v^{\sigma(p+1)}, \ldots, v^{\sigma(p+q)})\gamma(v^{\tau(p+q+1)}, \ldots, v^{\tau(p+q+r)}),
\]
where \((\sigma_\tau(1), \ldots, \sigma_\tau(p + q))\) is obtained by applying the permutation \( \sigma \) to \((\tau(1), \ldots, \tau(p + q))\). Therefore
\[
(\alpha \wedge \beta) \wedge \gamma(v^1, \ldots, v^{p+q+r})
\]
is a permutation of \((1, \ldots, p + q + r)\), which sign exactly \(\text{sgn}(\tau)\text{sgn}(\sigma)\). Furthermore every element in \(S_{p+q+r}\) can be obtained in this way, in exactly \((p + q)!\) different ways. It follows that
\[
(\alpha \wedge \beta) \wedge \gamma(v^1, \ldots, v^{p+q+r})
= \frac{1}{p!q!r!} \sum_{\rho \in S_{p+q+r}} \sum_{\sigma \in S_{p+q}} \text{sgn}(\rho)\alpha(v^{\rho(1)}, \ldots, v^{\rho(p)})\beta(v^{\rho(p+1)}, \ldots, v^{\rho(p+q)})\gamma(v^{\rho(p+q+1)}, \ldots, v^{\rho(p+q+r)}),
\]
The same argument shows that this expression also equals \(\alpha \wedge (\beta \wedge \gamma)(v^1, \ldots, v^{p+q+r})\), which concludes the proof.

4) Let \((e_1, e_2, e_3)\) be the standard basis of \(\mathbb{R}^3\). Consider the isomorphism
\[
P : \Lambda^2 \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ defined by }
\]
\[
e_2 \wedge e_3 \mapsto e_1, \quad e_3 \wedge e_1 \mapsto e_2, \quad e_1 \wedge e_2 \mapsto e_3.
\]
Show that given any two vectors \(v, w \in \mathbb{R}^3 = \Lambda^1 \mathbb{R}^3\), \(P(v \wedge w)\) equals the cross product \(v \times w\).
**Solution.** Since both $P(v \wedge w)$ and $v \times w$ are bilinear, it is enough to check that they agree on all pairs $(e_i, e_j)$ of basis vectors. Since $e_i \wedge e_i = 0 = e_i \times e_i$ and $e_i \wedge e_j = -e_j \wedge e_i, e_i \times e_j = -e_j \times e_i$, we only need to consider pairs $(e_i, e_j)$ with $i < j$. There are three such pairs, and the two maps agree on these pairs thanks to the definitions of $P$ and $\times$. 