1) Let $X_{p,q}$ be the complex manifold diffeomorphic to $S^{2p-1} \times S^{2q-1}$ which you constructed in Homework 1, problem 1. For which values of $p, q$ is $X_{p,q}$ Kähler?

**Solution.** When $p = q = 1$ we get $X_{1,1} = S^1 \times S^1$ which is a Riemann surface and therefore Kähler. When $p$ or $q$ is bigger than 1, we have that

$$H^2(X_{p,q}, \mathbb{R}) = H^2(S^{2p-1}, \mathbb{R}) \oplus H^2(S^{2q-1}, \mathbb{R}) \oplus (H^1(S^{2p-1}, \mathbb{R}) \otimes H^1(S^{2q-1}, \mathbb{R})) = 0,$$

by the Künneth formula, and so $X_{p,q}$ cannot be Kähler.

2) Let $Y = X/\sigma$ be the complex manifold constructed in Homework 1, problem 3.

(a) Show that $Y$ is Kähler.

(b) Calculate the Hodge numbers $h^{1,0}(Y), h^{0,1}(Y), h^{2,0}(Y), h^{1,1}(Y)$ and $h^{0,2}(Y)$.

**Solution.** (a) The flat Kähler form $\omega = idz \wedge d\overline{z} + idw \wedge d\overline{w}$ on $\mathbb{C}^2$ satisfies $\sigma^* \omega = \omega$, and so it descends to a flat Kähler metric on $Y$.

(b) We know from Homework 1, problem 3 (b), that $h^{2,0}(Y) = 0$, and by Serre duality it follows that $h^{0,2}(Y) = 0$. Since $Y$ is Kähler by (a), we deduce that

$$h^{1,1}(Y) = b_2(Y), \quad h^{1,0}(Y) = h^{0,1}(Y) = \frac{b_1(Y)}{2},$$

so it is enough to compute the Betti numbers of $Y$. Let’s calculate $H^1(Y, \mathbb{R})$ first. This is just given by the deRham cohomology classes of $X$ which are invariant under $\sigma$. A basis of $H^1(X, \mathbb{R})$ is given by the classes of the forms $dx, dy, ds, dt$, and since

$$\sigma^* dx = dx, \quad \sigma^* dy = dy, \quad \sigma^* ds = -ds, \quad \sigma^* dt = -dt,$$

we see that among these only $dx, dy$ are invariant under $\sigma$, therefore $b_1(Y) = 2$ and $h^{1,0}(Y) = h^{0,1}(Y) = 1$. Similarly, a basis for $H^2(X, \mathbb{R})$ is given by the
classes of the forms
\[ dx \wedge dy, \ dx \wedge ds, \ dx \wedge dt, \ dy \wedge ds, \ dy \wedge dt, \ ds \wedge dt, \]
and among these only \( dx \wedge dy, \ ds \wedge dt \) are invariant under \( \sigma \), therefore \( b_2(Y) = 2 \) and \( h^{1,1}(Y) = 2 \).

3) Let \( X \) be a compact complex manifold. Prove that the map that associates to a smooth real 1-form its \((0,1)\) part induces an injective map
\[ H^1(X, \mathbb{R}) \hookrightarrow H^{0,1}_\partial(X), \]
from deRham cohomology to Dolbeault cohomology. Show that this implies that \( b_1(X) \leq 2h^{0,1}(X) \).

Solution: We start with the map that associates to a real 1-form \( a \) its \((0,1)\)-part \( a^{0,1} \). We have \( a = a^{0,1} + a^{0,1} \). If \( da = 0 \), then
\[ 0 = \partial a^{0,1} + \bar{\partial} a^{0,1} + \bar{\partial} a^{0,1} + \partial a^{0,1}, \]
and taking the \((0,2)\)-part of this identity we see that \( \bar{\partial} a^{0,1} = 0 \). Therefore we get a map from the space of \( d \)-closed real 1-forms to \( H^{0,1}_\partial(X) \). If \( a = db \) is \( d \)-exact, then \( a^{0,1} = \bar{\partial} b \) is \( \bar{\partial} \)-exact, and so we get a linear map \( H^1(X, \mathbb{R}) \rightarrow H^{0,1}_\partial(X) \). To see that this is injective, assume that \( a \) is in its kernel, so \( a^{0,1} = \bar{\partial} \beta \) for some smooth complex-valued function \( \beta \). Then
\[ a = a^{0,1} + a^{0,1} = \bar{\partial} \beta + \bar{\partial} \beta = d(\beta + \bar{\beta}) - \partial \beta - \bar{\partial} \beta. \]
Take the exterior derivative of this equation and get
\[ 0 = da = -\bar{\partial} \partial \beta - \partial \bar{\partial} \beta = \bar{\partial}(\beta - \bar{\beta}) = 2\sqrt{-1} \partial \bar{\partial} \text{Im}(\beta), \]
so the real function \( \text{Im}(\beta) \) is pluriharmonic and therefore by the strong maximum principle it is constant. Therefore its derivatives are zero, which implies that \( \partial(\beta - \bar{\beta}) = 0 \), or in other words \( \partial \beta = \partial \bar{\beta} \). Substituting this in the above equation we get
\[ a = \partial \bar{\beta} + \bar{\partial} \beta = \partial \left( \frac{\beta + \bar{\beta}}{2} \right) + \bar{\partial} \left( \frac{\beta + \bar{\beta}}{2} \right) = d(\text{Re}(\beta)), \]
so \( a \) is \( d \)-exact, and the map is injective. Therefore
\[ b_1(X) = \dim_{\mathbb{R}} H^1(X, \mathbb{R}) \leq \dim_{\mathbb{R}} H^{0,1}_\partial(X) \]
\[ = 2 \dim_{\mathbb{C}} H^{0,1}_\partial(X) = 2h^{0,1}(X). \]