1) Let $F: M \to N$ be a smooth map between compact oriented $n$-manifolds, with orientations determined by volume forms $\nu_M \in \Lambda^n(M)$ and $\nu_N \in \Lambda^n(N)$. The degree of $F$ is the real number defined by the identity

$$\int_M F^*\nu_N = (\deg F) \int_N \nu_N.$$ 

(a) Show that $\deg F$ does not depend on the choice of volume forms (of course, it depends on $F$ and on the chosen orientations).

(b) Show that if $G: M \to N$ is a smooth map which is homotopic to $F$, then $\deg F = \deg G$.

**Solution.** (a) The degree is clearly independent of $\nu_M$, since only its given orientation is needed to define the integral $\int_M F^*\nu_N$. We now check that it does not depend on the choice of $\nu_N$. Consider the pullback map $F^*: H^n(N) \to H^n(M)$. We have that $H^n(N) \cong \mathbb{R} \cong H^n(M)$, and if $[\alpha]$ is a generator of $H^n(N)$ and $[\beta]$ is a generator of $H^n(M)$, then $F^*[\alpha] = \lambda[\beta]$ for some $\lambda \in \mathbb{R}$. Thanks to Poincaré duality, $\int_N[\alpha] \neq 0$, $\int_N[\beta] \neq 0$, and so up to scaling $[\alpha], [\beta]$ we may assume that these integrals are both equal to 1. We claim that then $\deg F = \lambda$.

Indeed, if $\nu_N$ is a volume form on $N$ compatible with the given orientation, then $[\nu_N] = \mu[\alpha]$ for some $\mu \neq 0$ (since $[\nu_N] \neq 0$ in $H^n(N)$), and so

$$(\deg F)\mu = (\deg F) \int_N \nu_N = \int_M F^*\nu_N = \mu \int_M F^*[\alpha] = \lambda \mu.$$ 

(b) This follows immediately from the description given in (a) of the degree of a map, together with the fact that homotopic maps $F, G$ have the same pullbacks $F^*, G^*$ in cohomology.

2) In the same setup as problem 1, let $y \in N$ be a regular value for $F$, and let $x \in F^{-1}(y)$. Define $\varepsilon(x) = 1$ if $dF_x$ is orientation preserving and $\varepsilon(x) = -1$ otherwise.

(a) Show that

$$\deg F = \sum_{x \in F^{-1}(y)} \varepsilon(x),$$

independent of the regular value $y \in F(M)$.
(b) Deduce that $\deg F \in \mathbb{Z}$

(c) Show that if $F$ is not surjective then $\deg F = 0$

(d) Show that if $F$ is a diffeomorphism then $\deg F = \pm 1$, depending on whether $F$ preserves or reverses the orientation.

**Solution.** (a) Since $y$ is a regular value and $M, N$ are compact manifolds of the same dimension, it follows that $F^{-1}(y)$ is a finite number of points \( \{x_1, \ldots, x_N\} \) (since it is a compact 0-submanifold of $M$) and near each $x_j \in F^{-1}(y)$ the map $F$ is a local diffeomorphisms. In other words, there is a connected open subset $U \ni y$ of $N$ with $F^{-1}(U) = \bigcup_{j=1}^{N} U_j$, with each $U_j \ni x_j$ connected and $F : U_j \to U$ is a diffeomorphism. Therefore $F : U \ni U$ is orientation preserving iff $\varepsilon(x_j) = 1$ and orientation reversing iff $\varepsilon(x_j) = -1$. Let $\alpha$ be a smooth $n$-form on $N$ supported in $U$ with $\int_N \alpha = 1$. Then $[\alpha]$ is a generator of $H^n(N)$. Thanks to the solution of problem 1, we have that

$$\deg F = (\deg F) \int_N \alpha = \int_M F^* \alpha = \sum_j \int_{U_j} F^* \alpha = \sum_j \varepsilon(x_j) \int_U \alpha = \sum_j \varepsilon(x_j).$$

(b) This is trivial from part (a).

(c) If $F$ is not surjective, then there is a point $y \in N$ not in the image of $F$. Since $F(M)$ is closed in $N$, there is an open subset $U \ni y$ of $N$ which does not intersect $F(M)$. Let $\alpha$ be a smooth $n$-form on $N$ supported in $U$ with $\int_N \alpha = 1$. Then $[\alpha]$ is a generator of $H^n(N)$. Thanks to the solution of problem 1, we have that

$$\int_M F^* \alpha = (\deg F) \int_N \alpha = \deg F.$$

However since $F(M)$ does not intersect the support of $\alpha$, $\int_M F^* \alpha = 0$.

(d) This was proved in class, since we showed that if $F$ is a diffeomorphism then

$$\int_M F^* \nu_N = \pm \int_N \nu_N,$$

depending on whether $F$ preserves or reverses the orientation.

3) Let $F, G : S^n \to S^n$ be smooth maps.

(a) Show that if $F(x) \neq -G(x)$ for all $x \in S^n$, then $F$ and $G$ are homotopic

(b) Show that if $F(x) \neq -x$ for all $x \in S^n$, then $\deg F = 1$
(c) Show that if \( F(x) \neq x \) for all \( x \in S^n \), then \( \deg F = (-1)^{n+1} \)

(d) Use (b) and (c) to show that if \( n \) is even then every smooth vector field on \( S^n \) has a zero.

**Solution.**

(a) Given \( x \in S^n \), since \( F(x) \neq -G(x) \), we see that the segment 
\[
(1-t)F(x) + tG(x), 0 \leq t \leq 1,
\]
joining \( F(x) \) and \( G(x) \) does not pass through the origin. Therefore
\[
H(t, x) = \frac{(1-t)F(x) + tG(x)}{|(1-t)F(x) + tG(x)|}, 0 \leq t \leq 1,
\]
is a homotopy between \( F \) and \( G \) (here \( | \cdot | \) is the Euclidean norm on \( \mathbb{R}^{n+1} \), and we have embedded \( S^n \subset \mathbb{R}^{n+1} \) as the unit sphere).

(b) Take \( G = \text{Id} \) in part (a). Thanks to problem 1, we see that \( \deg F = \deg \text{Id} = 1 \).

(c) Take \( G = A \), the antipodal map \( A(x) = -x \), in part (a). Thanks to problem 1, we see that \( \deg F = \deg A \), and we computed in class that \( \deg A = (-1)^{n+1} \).

(d) Let \( X \in T S^n \) be a smooth vector field which is never zero. Embed \( S^n \subset \mathbb{R}^{n+1} \) as the unit sphere, and for any \( x \in S^n \) regard \( X(x) \) as a vector in \( \mathbb{R}^{n+1} \). Then
\[
x \mapsto \frac{X(x)}{|X(x)|},
\]
defines a smooth map \( F : S^n \to S^n \). Since \( X(x) \) is tangent to \( S^n \), we have that \( X(x) \) is perpendicular to \( x \) (in \( \mathbb{R}^{n+1} \)) for all \( x \in X \). If we had \( F(x) = -x \) for some \( x \in S^n \), then taking the inner product with \( x \) we would conclude that \( x = 0 \), absurd. Similarly, \( F(x) \neq x \) for all \( x \in S^n \). From parts (b) and (c) we see that
\[
1 = \deg F = (-1)^{n+1},
\]
which implies that \( n \) is odd, as required.