1) Consider the standard covering \( \mathcal{U} = \{ U_0, U_1 \} \) of \( \mathbb{C}P^1 \), where in homogeneous coordinates \([ z_0 : z_1 ]\) we have set \( U_j = \{ z_j \neq 0 \}, j = 0, 1 \), with local holomorphic coordinates \( u = \frac{z_1}{z_0} \) on \( U_0 \cong \mathbb{C} \) and \( v = \frac{z_0}{z_1} \) on \( U_1 \cong \mathbb{C} \). On the overlap \( U_0 \cap U_1 \cong \mathbb{C}^* \) these two coordinates are related by \( v = \frac{1}{u} \). Using the definition, compute the Čech cohomology groups \( H^0(\mathcal{U}, \mathcal{O}) \) and \( H^1(\mathcal{U}, \mathcal{O}) \).

2) Let \( X \) be a compact complex manifold of (complex) dimension \( n \).

   (a) Show that if \( \eta \) is a holomorphic \( n \)-form on \( X \) which is \( d \)-exact, then \( \eta \) is identically zero.

   (b) Show that if \( \gamma \) is any holomorphic \( n-1 \) form on \( X \), then \( \gamma \) is \( d \)-closed.

3) Let

\[
G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{C} \right\},
\]

which is a complex Lie subgroup of \( SL(3, \mathbb{C}) \), biholomorphic to \( \mathbb{C}^3 \), and let

\[
\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{Z}[i] \right\},
\]

be the subgroup where \( x, y, z \) are Gaussian integers. \( \Gamma \) acts on \( G \) by left multiplication of matrices, the action is free and properly discontinuous, and the quotient \( X = G/\Gamma \) is a compact complex 3-fold (you don’t need to prove these).

   (a) Show that the holomorphic 1-forms \( dx, dy \) and \( dz - xdy \) on \( G \) are \( \Gamma \)-invariant and descend to global holomorphic 1-forms on \( X \).

   (b) Conclude that \( X \) admits a holomorphic 1-form which is not \( d \)-closed. Compare this with problem 2(b).

4) Let \( X \) be a complex manifold, and \( \alpha \) a \( (1, 1) \) form on \( X \) which is real (i.e. \( \alpha = \bar{\alpha} \)) and \( d \)-closed. Show that for every \( x \in X \) there exist an open set \( U \ni x \) and a smooth real-valued function \( f \) on \( U \) such that we have

\[
\alpha = i\partial\bar{\partial}f,
\]
on \( U \).