In this homework, you may only use the tools that we developed in class so far (e.g. you cannot use Riemann-Roch).

1) Let $X$ be a compact Riemann surface, $p \in X$ a given point and let $D = \{p\}$, the divisor corresponding to $p$.

(a) Calculate $\dim \mathbb{C} H^0(X, \mathcal{O}(D))$ when $X = \mathbb{CP}^1$.

(b) Calculate $\dim \mathbb{C} H^0(X, \mathcal{O}(D))$ when $X$ has genus large than or equal to 1.

**Solution.** (a) We know that $H^0(X, \mathcal{O}(D))$ is identified with the vector space of meromorphic functions on $X$ with at worst a simple pole at $p$. Constant functions are always in this space. Also, given a nontrivial principal part at $p$, say $\frac{1}{z}$, the obstruction to the existence of a global meromorphic function with this prescribed principal part lies in $H^0 \partial (X)$ which in this case is zero, as mentioned in class. Therefore there is a non-constant meromorphic function $f$ with only a simple pole at $p$, and hence everywhere (c is the quotients of the residues of $f$ and $g$ in a coordinate chart near $p$). Since $X$ is compact, it follows that $f - cg$ is constant, and therefore $\dim \mathbb{C} H^0(X, \mathcal{O}(D)) = 2$ (spanned by 1 and $f$).

(b) Again constants are always in $H^0(X, \mathcal{O}(D))$. Suppose there exists $f$ a non-constant meromorphic function on $X$ with precisely a simple pole at $p$ (and no other poles). Then $f$ defines a holomorphic map $F : X \to \mathbb{CP}^1$. Recall that for any nontrivial meromorphic function on a compact Riemann surface, the number of zeros (with multiplicites) equals the number of poles (with multiplicities). Since for any $\lambda \in \mathbb{C}$, $f - \lambda$ has a unique simple pole, it follows that it has a unique simple zero. Similarly, the fact that $f$ has a unique pole means that there is only one point in the preimage $F^{-1}(\infty)$, where $\infty \in \mathbb{CP}^1$. This implies that $F$ is bijective, and hence a biholomorphism. This contradicts the assumption that the genus of $X$ is at least 1, and so proves that $H^0(X, \mathcal{O}(D))$ consists only of constants, and so $\dim \mathbb{C} H^0(X, \mathcal{O}(D)) = 1$.

2) Let $X$ be a compact Riemann surface which has a holomorphic line bundle $L$ with $\deg L = 1$ and $\dim \mathbb{C} H^0(X, L) = 2$. Prove that $X$ is biholomorphic to $\mathbb{CP}^1$.

**Solution.** Let $\{s_1, s_2\}$ be a basis of $H^0(X, L)$. Since $s_1$ is holomorphic and the divisors of its zeros has degree 1, it follows that $s_1$ has a unique zero (which is simple). The same of course applies to $s_2$. Furthermore, the zeros of $s_1$ and $s_2$ must be distinct, otherwise $s_1/s_2$ would define a global holomorphic function on $X$, which must be constant, violating the linear independence of $s_1$ and $s_2$. Therefore the ratio $f = s_1/s_2$ defines a global meromorphic function on $X$ with precisely one pole (which is simple). Then the same argument as in problem 1 shows that $f$ defines a biholomorphism between $X$ and $\mathbb{CP}^1$.

3) Let $X$ be a compact complex manifold and $L$ a $C^\infty$ complex line bundle on $X$. Let $\mathcal{E}$ denote the sheaf of germs of smooth complex-valued functions on $X$, and by $\mathcal{E}^*$ those which
never vanish. Using the commutative diagram of sheaves on $X$ (with exact rows)

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^* & \rightarrow & 0 \\
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathcal{E} & \xrightarrow{\exp} & \mathcal{E}^* & \rightarrow & 0
\end{array}
$$

where $\exp$ is the map $\exp(2\pi i \cdot)$, show that $c_1(L) \in H^2(X, \mathbb{Z})$ is a torsion element (equivalently, $c_1(L) = 0$ in $H^2(X, \mathbb{R})$) if and only if $L$ is $C^\infty$ isomorphic to a line bundle whose transition functions (on some trivializing cover) are constant functions (with values in $\mathbb{C}^*$, of course).

**Solution.** As we know, the space of isomorphism classes of smooth complex line bundles is identified with $H^1(X, \mathcal{E}^*)$. The snake lemma gives us the commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
H^1(X, \mathbb{C}) & \xrightarrow{\alpha} & H^1(X, \mathbb{C}^*) & \xrightarrow{\beta} & H^2(X, \mathbb{Z}) & \xrightarrow{\gamma} & H^2(X, \mathbb{C}) \\
\downarrow{\varepsilon} & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^1(X, \mathcal{E}^*) & \xrightarrow{c_1} & H^2(X, \mathbb{Z}) & \rightarrow & 0
\end{array}
$$

where the two zeros on the bottom row come from the fact that $H^1(X, \mathcal{E}) = H^2(X, \mathcal{E}) = 0$ since $\mathcal{E}$ is a fine sheaf. In particular, the map $c_1$ is an isomorphism.

We have that $L$ has constant transition functions (with values in $\mathbb{C}^*$) on some trivializing cover, if and only if $L \in H^1(X, \mathcal{E}^*)$ (more precisely, its isomorphism class) lies in the image of the map $\varepsilon$. One direction is now easy: if $L$ has constant transition functions, then $L = \varepsilon(A), A \in H^1(X, \mathbb{C}^*)$, and $\gamma(\beta(A)) = 0$ by exactness, but by commutativity this equals $\gamma(c_1(\varepsilon(A))) = \gamma(c_1(L))$. So $\gamma(c_1(L)) = 0$, which by definition is equivalent to $c_1(L)$ being torsion in $H^2(X, \mathbb{Z})$.

Conversely, if $\gamma(c_1(L)) = 0$, then by exactness there exists $A \in H^1(X, \mathbb{C}^*)$ such that $\beta(A) = c_1(L)$. By commutativity, we get $c_1(\varepsilon(A) - L) = 0$, and so $L = \varepsilon(A)$, as required.

4) Let $X$ be a complex manifold and $\mathcal{P}$ be the sheaf given by

$$
\mathcal{P}(U) = \{f : U \rightarrow \mathbb{R} \mid i\partial\bar{\partial}f = 0\}
$$

i.e. $\mathcal{P}$ is the sheaf of pluriharmonic functions. Prove that the diagram of sheaves on $X$

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \mathbb{R} & \rightarrow & \mathcal{O} & \xrightarrow{\text{Im}} & \mathcal{P} & \rightarrow & 0 \\
\downarrow{q} & & \downarrow{\exp} & & \downarrow & & \\
0 & \rightarrow & \mathbb{R}/\mathbb{Z} & \xrightarrow{i} & \mathcal{O}^* & \xrightarrow{\ell} & \mathcal{P} & \rightarrow & 0
\end{array}
$$

is commutative with exact rows, where $q$ is the quotient map, $\ell(f) = -\frac{i}{2\pi} \log |f|$, $i$ is induced by the inclusion of $\mathbb{R}/\mathbb{Z}$ in $\mathbb{C}^*$ as the unit circle, and $\text{Im}$ is the imaginary part.

**Solution.** First, we check exactness of the first row. The inclusion of sheaves $\mathbb{R} \subset \mathcal{O}$ is obvious. The surjectivity of the map $\text{Im}$ was proved in class: every pluriharmonic function
is locally the imaginary part of a holomorphic function. Exactness at $\mathcal{O}$ follows from the fact that if a holomorphic function is real-valued then it is locally constant.

Second, we check exactness of the second row. Again the inclusion is obvious. The surjectivity of $\ell$ is done as follows: if $u$ is pluriharmonic, locally $u = \text{Im} f$ for $f$ holomorphic, and so $u = \ell(\text{e}^{2\pi i f})$ and $\text{e}^{2\pi i f}$ is holomorphic and never zero. Exactness at $\mathcal{O}^*$ follows from the fact that if a holomorphic function has constant modulus then it is locally constant.

Third, we check commutativity of the first square. This just follows from the fact that $\text{e}^{2\pi i \theta}$ parametrizes the unit circle in $\mathbb{C}$, as $0 \leq \theta \leq 1$.

Lastly, we check commutativity of the second square. This just follows from the identity $\ell(\exp(f)) = \text{Im}(f)$ for any $f$ holomorphic.