You can use freely the following simple results from complex analysis (which can be proved in the same way as the case of one complex variable). Recall that $\mathbb{C}P^n = (\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts by scalar multiplication. Let $(z_0, \ldots, z_n)$ be the coordinates on $\mathbb{C}P^n$, and write $z_j = x_j + iy_j$, $0 \leq j \leq n$, with $(x_0, \ldots, y_n)$ real coordinates on $\mathbb{C}^{n+1}$. We shall use complex derivatives

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right),$$

and complex 1-forms $dz_j = dx_j + idy_j$, $d\bar{z}_j = dx_j - idy_j$. Then for any smooth function $f$ we have

$$df = \sum_{j=0}^n \frac{\partial f}{\partial z_j} dz_j + \sum_{j=0}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

We also have that

$$\frac{\partial z_j}{\partial z_k} = \delta_{jk}, \quad \frac{\partial z_j}{\partial \bar{z}_k} = 0.$$

We will write $|z|^2 = \sum_{j=0}^n |z_j|^2$.

1) On $\mathbb{C}^{n+1}\setminus\{0\}$ let

$$g_{jk} = \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_k} \log |z|^2.$$

Prove that the complex 2-form

$$\omega = i \sum_{j,k=0}^n g_{jk} dz_j \wedge d\bar{z}_k,$$

on $\mathbb{C}^{n+1}\setminus\{0\}$ is real (i.e. $\omega = \bar{\omega}$), closed and invariant under the $\mathbb{C}^*$ action. Denote by $\omega_{FS}$ the closed real 2-form it induces on $\mathbb{C}P^n$.

2) In the chart $U_0 = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid z_0 \neq 0\} \cong \mathbb{C}^n$ with coordinates $w_j = z_j/z_0$, $j = 1, \ldots, n$, show that

$$\omega^n_{FS} = \frac{n!}{(1 + |w|^2)^{n+1}} idw_1 \wedge d\overline{w}_1 \wedge \cdots \wedge idw_n \wedge d\overline{w}_n.$$

3) Again using the chart $U_0$, compute

$$\int_{\mathbb{C}P^n} \omega^n_{FS} = (2\pi)^n.$$

Deduce from this that $[\omega_{FS}^k] \neq 0$ in $H^{2k}(\mathbb{C}P^n)$ for all $0 \leq k \leq n$, and show that therefore $[\omega_{FS}^k]$ is a generator of $H^{2k}(\mathbb{C}P^n)$.

4) Use the Lefschetz fixed point theorem and show that for any even $n$, every smooth map $F : \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point.

5) For any odd $n$, find a smooth map $F : \mathbb{C}P^n \to \mathbb{C}P^n$ without fixed points.