You can use freely the following simple results from complex analysis (which can be proved in the same way as the case of one complex variable). Recall that $\mathbb{CP}^n = (\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts by scalar multiplication. Let $(z_0, \ldots, z_n)$ be the coordinates on $\mathbb{C}^{n+1}$, and write $z_j = x_j + iy_j$, $0 \leq j \leq n$, with $(x_0, \ldots, y_n)$ real coordinates on $\mathbb{C}^{n+1}$. We shall use complex derivatives

$$
\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right),
$$

and complex 1-forms $dz_j = dx_j + idy_j, d\bar{z}_j = dx_j - idy_j$. Then for any smooth function $f$ we have

$$
df = \sum_{j=0}^{n} \frac{\partial f}{\partial z_j} dz_j + \sum_{j=0}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.
$$

We also have that

$$
\frac{\partial z_j}{\partial \bar{z}_k} = \delta_{jk}, \quad \frac{\partial z_j}{\partial z_k} = 0.
$$

We will write $|z|^2 = \sum_{j=0}^{n} |z_j|^2$.

1) On $\mathbb{C}^{n+1}\setminus\{0\}$ let

$$
g_{jk} = \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_k} \log |z|^2.
$$

Prove that the complex 2-form

$$
\omega = i \sum_{j,k=0}^{n} g_{jk} dz_j \wedge d\bar{z}_k,
$$

on $\mathbb{C}^{n+1}\setminus\{0\}$ is real (i.e. $\omega = \overline{\omega}$), closed and invariant under the $\mathbb{C}^*$ action. Denote by $\omega_{FS}$ the closed real 2-form it induces on $\mathbb{CP}^n$.

**Solution.** First of all we claim that $\overline{g_{jk}} = g_{kj}$, i.e. that at each point the matrix $g_{jk}$ is Hermitian. This is clear, because $\log |z|^2$ is a real-valued function and $\frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_k} = \frac{\partial}{\partial \bar{z}_k} \frac{\partial}{\partial z_j}$. Therefore

$$
\overline{\omega} = -i \sum_{j,k=0}^{n} g_{kj} d\bar{z}_j \wedge dz_k = i \sum_{j,k=0}^{n} g_{kj} dz_k \wedge d\bar{z}_j = \omega,
$$
which means that \( \omega \) is real (and after switching to real coordinates, it becomes a usual real 2-form). To show that \( \omega \) is closed, we compute

\[
d\omega = i \sum_{j,k} (dg_{jk}) \wedge dz_j \wedge d\overline{z}_k = i \sum_{j,k,\ell} \frac{\partial}{\partial z_\ell} \frac{\partial}{\partial z_j} \frac{\partial}{\partial \overline{z}_k} \log |z|^2 dz_\ell \wedge dz_j \wedge d\overline{z}_k
\]

\[
+ i \sum_{j,k,\ell} \frac{\partial}{\partial \overline{z}_\ell} \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_k} \log |z|^2 d\overline{z}_\ell \wedge dz_j \wedge d\overline{z}_k = 0,
\]

because partial derivatives commute. Let us now compute

\[
g_{jk} = \frac{\partial}{\partial z_j} \left( \frac{z_k}{|z|^2} \right) = \frac{|z|^2 \delta_{jk} - z_k z_j}{|z|^4}.
\]

If \( \lambda \in \mathbb{C}^* \) and \( F_\lambda : \mathbb{C}^{n+1}\{0\} \to \mathbb{C}^{n+1}\{0\} \) is given by \( F_\lambda(z) = \lambda z \) then

\[
F_\lambda^* \omega = i|\lambda|^2 \sum_{j,k=0}^n (g_{jk} \circ F_\lambda) dz_j \wedge d\overline{z}_k = \omega.
\]

2) In the chart \( U_0 = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid z_0 \neq 0\} \cong \mathbb{C}^n \) with coordinates \( w_j = z_j/z_0, j = 1, \ldots, n \), show that

\[
\omega_{FS}^n = \frac{n!}{(1 + |w|^2)^{n+1}} idw_1 \wedge d\overline{w}_1 \wedge \cdots \wedge idw_n \wedge d\overline{w}_n.
\]

**Solution.** On the subset of \( \mathbb{C}^{n+1}\{0\} \) where \( z_0 \neq 0 \), we have

\[
\log |z|^2 = \log |z_0|^2 + \log(1 + |w|^2),
\]

and

\[
\frac{\partial}{\partial z_j} \frac{\partial}{\partial \overline{z}_k} \log |z_0|^2 = \frac{|z_0|^2 - |z_0|^2}{|z_0|^4} = 0.
\]

It follows that on \( U_0 \) we can write

\[
\omega_{FS} = i \sum_{j,k=1}^n (1 + |w|^2) \delta_{jk} - w_k \overline{w}_j \frac{d w_j \wedge d \overline{w}_k}{(1 + |w|^2)^2}.
\]

Let us write

\[
h_{jk} = \frac{(1 + |w|^2) \delta_{jk} - w_k \overline{w}_j}{(1 + |w|^2)^2},
\]
which is a Hermitian $n \times n$ matrix at each point. If we apply a unitary change of coordinates, we may diagonalize $h_{jk}$, so that in these new coordinates $\zeta_j$ we have
\[
\omega_{FS} = i \sum_{j=1}^{n} \lambda_j d\zeta_j \wedge d\overline{\zeta}_j,
\]
with $\lambda_j \in \mathbb{R}$. Taking the $n^{th}$ wedge product of this form, we get
\[
\omega^n_{FS} = n! \left( \prod_{j=1}^{n} \lambda_j \right) id\zeta_1 \wedge d\overline{\zeta}_1 \wedge \cdots \wedge id\zeta_n \wedge d\overline{\zeta}_n,
\]
and switching back to the original coordinates,
\[
\omega^n_{FS} = n! \det(h_{jk}) idw_1 \wedge d\overline{w}_1 \wedge \cdots \wedge idw_n \wedge d\overline{w}_n,
\]
and we only need to compute the determinant of $h_{jk}$. This equals
\[
\det(h_{jk}) = \frac{1}{(1 + |w|^2)^{2n}} \det((1 + |w|^2) \delta_{jk} - w_k \overline{w}_j).
\]

The Hermitian matrix $\frac{w_k \overline{w}_j}{|w|^n}$ represents the rank 1 linear transformation which is the projection onto the complex line spanned by the nonzero complex number $w$, and therefore the matrix $w_k \overline{w}_j$ has the eigenvalue $|w|^2$ with multiplicity 1 and the eigenvalue 0 with multiplicity $n-1$. Since $(1 + |w|^2) \delta_{jk}$ is a multiple of the identity (which remains such in all coordinate systems, for example in coordinates which diagonalize $w_k \overline{w}_j$), we see that the matrix $(1 + |w|^2) \delta_{jk} - w_k \overline{w}_j$ has eigenvalue 1 with multiplicity 1 and eigenvalue $(1 + |w|^2)$ with multiplicity $n-1$. It follows that
\[
\det((1 + |w|^2) \delta_{jk} - w_k \overline{w}_j) = (1 + |w|^2)^{n-1},
\]
and so
\[
\det(h_{jk}) = \frac{1}{(1 + |w|^2)^{n+1}},
\]
as required.

3) Again using the chart $U_0$, compute
\[
\int_{\mathbb{C}P^n} \omega^n_{FS} = (2\pi)^n.
\]
Deduce from this that $[\omega^k_{FS}] \neq 0$ in $H^{2k}(\mathbb{CP}^n)$ for all $0 \leq k \leq n$, and show that therefore $[\omega^k_{FS}]$ is a generator of $H^{2k}(\mathbb{CP}^n)$.

**Solution.** Note that

$$idw_j \wedge d\overline{w}_j = 2dx_j \wedge dy_j,$$

so that

$$idw_1 \wedge d\overline{w}_1 \wedge \cdots \wedge idw_n \wedge d\overline{w}_n = 2^n d\mu,$$

where $d\mu$ is the usual Lebesgue measure in $\mathbb{C}^n$. Since $\mathbb{CP}^n \setminus U_0 \equiv \mathbb{CP}^{n-1}$ has measure zero, we see that

$$\int_{\mathbb{CP}^n} \omega^n_{FS} = 2^n n! \int_{\mathbb{C}^n} \frac{1}{(1 + |w|^2)^{n+1}} d\mu(w).$$

We write $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$, with $w = (w', w_n)$, and first integrate in the variable $w_n$. We have

$$\int_{\mathbb{C}^n} \frac{1}{(1 + |w|^2)^{n+1}} d\mu(w) = \int_{\mathbb{C}^{n-1}} \int_{\mathbb{C}} \frac{1}{(1 + |w'|^2 + |w_n|^2)^{n+1}} d\mu(w_n) d\mu(w').$$

Use polar coordinates in $\mathbb{C}$ and write the innermost integral as

$$2\pi \int_0^\infty \frac{r}{(1 + |w'|^2 + r^2)^{n+1}} dr = \pi \int_0^\infty \frac{1}{(1 + |w'|^2 + s)^{n+1}} ds = \frac{\pi}{n(1 + |w'|^2)^n},$$

so that

$$\int_{\mathbb{C}^{n-1}} \frac{1}{(1 + |w'|^2)^{n+1}} d\mu(w') = \frac{n}{\pi} \int_{\mathbb{C}^{n-1}} \frac{1}{(1 + |w'|^2)^{n+1}} d\mu(w'),$$

which can be iterated to get

$$\int_{\mathbb{C}^n} \frac{1}{(1 + |w|^2)^{n+1}} d\mu(w) = \frac{n^n}{n!},$$

and so

$$\int_{\mathbb{CP}^n} \omega^n_{FS} = 2^n n! \frac{n^n}{n!} = (2\pi)^n.$$

If we had $[\omega^k_{FS}] = 0$ in $H^{2k}(\mathbb{CP}^n)$ for some $0 \leq k \leq n$, then by taking the cup product with $[\omega^{n-k}_{FS}]$ we would get that also $[\omega^n_{FS}] = 0$ in $H^{2n}(\mathbb{CP}^n)$. By Stokes’s Theorem this would imply that

$$\int_{\mathbb{CP}^n} \omega^n_{FS} = 0,$$

a contradiction. Since we computed in Homework 2 that $H^{2k}(\mathbb{CP}^n) \cong \mathbb{R}$ for $0 \leq k \leq n$, it follows that any nonzero element in $H^{2k}(\mathbb{CP}^n)$ is a generator,
so in particular this holds for $[\omega_{FS}^k]$.

4) Use the Lefschetz fixed point theorem and show that for any even $n$, every smooth map $F : \mathbb{CP}^n \to \mathbb{CP}^n$ has a fixed point.

**Solution.** Since $[\omega_{FS}]$ is a generator of $H^2(\mathbb{CP}^n)$, we have that $F^*[\omega_{FS}] = \lambda [\omega_{FS}]$ for some $\lambda \in \mathbb{R}$. Then $F^*[\omega_{FS}^k] = [(F^*[\omega_{FS}])^k] = \lambda^k [\omega_{FS}^k]$. But these are generators for the cohomology of $\mathbb{CP}^n$ (which vanishes in odd degree), and so by definition the Lefschetz number of $F$ is

$$L(F) = 1 + \lambda + \lambda^2 + \cdots + \lambda^n.$$ 

If $\lambda = 1$ then clearly $L(F) \neq 0$. If $\lambda \neq 1$, then $L(F) = (\lambda^{n+1} - 1)/(\lambda - 1)$. If this was zero, then we would have that $\lambda^{n+1} = 1$. If $n$ is even, this is impossible, since $\lambda$ is real. Therefore, if $n$ is even we conclude that $L(F) \neq 0$. Thanks to the Lefschetz fixed point theorem, $F$ has a fixed point.

5) For any odd $n$, find a smooth map $F : \mathbb{CP}^n \to \mathbb{CP}^n$ without fixed points.

**Solution.** We define $F : \mathbb{CP}^{2n+1} \to \mathbb{CP}^{2n+1}$ by

$$[z_0 : \cdots : z_{2n+1}] \mapsto [\bar{z}_1 : -\bar{z}_0 : \bar{z}_3 : -\bar{z}_2 : \cdots : -\bar{z}_{2n+1} : -\bar{z}_{2n}].$$

This is clearly a smooth map. If $[z_0 : \cdots : z_{2n+1}]$ was a fixed point for $F$, then we would have

$$z_0 = \lambda \bar{z}_1, \quad z_1 = -\lambda \bar{z}_0,$$

for some $\lambda \in \mathbb{C}^*$, and so $z_0 = -|\lambda|^2 z_0$, which forces $z_0 = z_1 = 0$. By the same token, $z_j = 0$ for all $j$, a contradiction.