Homework 5
Due February 9 in class.

The questions marked with * are a bit harder.

1) Let $F : M \to N$ be a smooth map between manifolds, and let $X, Y \in \mathcal{T}(N)$ be two vector fields on $N$ with the property that there are vector fields $\tilde{X}, \tilde{Y} \in \mathcal{T}(M)$ with $dF(\tilde{X}) = X, dF(\tilde{Y}) = Y$. Show that
\[ dF([\tilde{X}, \tilde{Y}]) = [X, Y]. \]

2) As usual identify $S^1 \subset \mathbb{C}$ with the set of unit complex numbers. In this way, the torus $S^1 \times S^1$ can be viewed as a subset of $\mathbb{C}^2$. For a given $\alpha \in \mathbb{R}$ let $\gamma_\alpha : \mathbb{R} \to S^1 \times S^1 \subset \mathbb{C}^2$ be given by
\[ \gamma_\alpha(t) = (e^{2\pi i t}, e^{2\pi i \alpha t}). \]

(a) If $\alpha \in \mathbb{Q}$ show that $\gamma_\alpha$ induces an embedding of $S^1$ into $S^1 \times S^1$.

(b)* If $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, show that $\gamma_\alpha$ is an injective immersion, and that its image is dense in $S^1 \times S^1$. Deduce that $\gamma_\alpha$ is not an embedding.

3) Determine explicitly the flow $\Theta$ of the vector field
\[ X = y \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \]
on $M = \mathbb{R}^2$.

4) Show that every $n$-manifold $M \ (n \geq 1)$ admits a diffeomorphism $F : M \to M$ which is not the identity.

5) Show that for every $n$-manifold $M \ (n \geq 1)$ the vector space $\mathcal{T}(M)$ of smooth vector fields on $M$ is infinite dimensional.