440-2 Geometry/Topology: Differentiable Manifolds
Northwestern University
Solution of Homework 5

The questions marked with * are a bit harder.

1) Let \( F : M \to N \) be a smooth map between manifolds, and let \( X, Y \in \mathcal{T}(N) \) be two vector fields on \( N \) with the property that there are vector fields \( \tilde{X}, \tilde{Y} \in \mathcal{T}(M) \) with \( dF(\tilde{X}) = X, dF(\tilde{Y}) = Y \). Show that \( dF([\tilde{X}, \tilde{Y}]) = [X, Y] \).

**Solution.** Given \( p \in M \), the equation \( dF_p(\tilde{X}) = X_{F(p)} \) is clearly equivalent to \( \tilde{X}(f \circ F) = X(f) \circ F \), for all germs \( f \in \mathcal{C}^\infty(F(p)) \), and similarly for \( \tilde{Y} \). Then we have \( \tilde{X}\tilde{Y}(f \circ F) = \tilde{X}(Y(f) \circ F) = (XY(f)) \circ F \), and so \( [\tilde{X}, \tilde{Y}](f \circ F) = ([X, Y](f)) \circ F \), which as we said is equivalent to \( dF_p([\tilde{X}, \tilde{Y}]) = [X, Y]_{F(p)} \).

2) As usual identify \( S^1 \subset \mathbb{C} \) with the set of unit complex numbers. In this way, the torus \( S^1 \times S^1 \) can be viewed as a subset of \( \mathbb{C}^2 \). For a given \( \alpha \in \mathbb{R} \), let \( \gamma_\alpha : \mathbb{R} \to S^1 \times S^1 \subset \mathbb{C}^2 \) be given by 
\[
\gamma_\alpha(t) = (e^{2\pi it}, e^{2\pi i\alpha t}).
\]

(a) If \( \alpha \in \mathbb{Q} \) show that \( \gamma_\alpha \) induces an embedding of \( S^1 \) into \( S^1 \times S^1 \).

(b)* Is \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), show that \( \gamma_\alpha \) is an injective immersion, and that its image is dense in \( S^1 \times S^1 \). Deduce that \( \gamma_\alpha \) is not an embedding.

**Solution.** (a) Clearly the image of \( \gamma_\alpha \) is contained in the torus \( S^1 \times S^1 \). As a curve in \( \mathbb{C}^2 \) we have 
\[
\gamma'_\alpha(t) = (2\pi ie^{2\pi it}, 2\pi i\alpha e^{2\pi i\alpha t}),
\]
which is never the zero vector, hence \( \gamma_\alpha \) is an immersion. If we write \( \alpha = p/q \) with \( p, q \in \mathbb{Z} \) coprime, \( q > 0 \) (if \( p = 0 \) we set \( q = 1 \)), then we see that for any given \( t \in \mathbb{R} \), \( \gamma_\alpha(t) = \gamma_\alpha(t + q) \), while \( \gamma_\alpha(t + s) \neq \gamma_\alpha(t) \) for \( 0 \leq s < q \).
Therefore $\gamma_\alpha$ descends to a smooth map from $S^1 = \mathbb{R} / (t \sim qt)$, which is an injective immersion. Its image is the compact subset $K$ of $S^1 \times S^1$ given by the equation $z_1^p = z_2^q$ where $(z_1, z_2) \in \mathbb{C}^2$. Indeed, the image is clearly contained in $K$, and conversely if $(z_1, z_2) \in \mathbb{C}^2$ satisfy $|z_1| = |z_2| = 1$ and $z_1^p = z_2^q$, write $z_1 = e^{2\pi ia}$, $z_2 = e^{2\pi ib}$, with $0 \leq a, b < 1$, then we must have $pa - qb = \ell \in \mathbb{Z}$. Since $p, q$ are coprime, we can add integers to $a, b$ so that $pa - qb = 0$, and then $\gamma_\alpha(a) = (z_1, z_2)$. As we vary $(z_1, z_2)$ continuously, we can choose $a$ which varies continuously as well, and this shows that the inverse of $\gamma_\alpha$, from $K$ to $S^1$ is continuous, hence $\gamma_\alpha$ is an embedding.

(b) We have seen in part (a) that $\gamma_\alpha$ is an injective immersion. Next we show that the image is dense in $S^1 \times S^1$. Consider the subset $\{e^{2\pi in}\}_{n \in \mathbb{Z}}$ of $S^1$. Since $S^1$ is compact, there is an accumulation point $z \in S^1$. Given $\varepsilon > 0$ choose integers $n_1 \neq n_2$ with $|e^{2\pi in} - 1| < \varepsilon/2$ for $j = 1, 2$. Then $k = n_1 - n_2$ is a nonzero integer which satisfies

$$|e^{2\pi ik} - 1| = |e^{2\pi i(n_1 - n_2)}| < \varepsilon.$$ 

Choose now an integer $m$ such that

$$0 < |ak - m| < \frac{1}{2},$$

which is possible (with strict inequalities) since $ak$ is irrational. We have $|e^{2\pi i(ak-m)} - 1| < \varepsilon$. The elementary inequality $|t| \leq |e^{2\pi it} - 1|$ (which holds for $0 \leq t \leq 1/2$), gives us $|ak - m| < \varepsilon$. Let now $p = (e^{2\pi ix}, e^{2\pi iy})$ be any point on $S^1 \times S^1$. For any $n \in \mathbb{Z}$, the Euclidean distance in $\mathbb{C}^2$ between $p$ and $\gamma_\alpha(x + n)$ is

$$|e^{2\pi i(x+n)} - e^{2\pi iy}| = |e^{2\pi i(b+n\alpha)} - 1|,$$

where $b \in [0, 1)$ is the fractional part of $ax - y$. If we can show that for any $\delta > 0$ there exists $n \in \mathbb{Z}$ such that $|e^{2\pi i(b+n\alpha)} - 1| < \delta$, then we will have proved that the image of $\gamma_\alpha$ is dense. If $b = 0$, then we have proved this before. If $b > 0$, let $h \in \mathbb{N}$ be such that

$$0 < h \leq \frac{b}{|ak - m|} < h + 1,$$

so

$$0 < b - h|ak - m| < |ak - m| < \varepsilon,$$

which we may assume is less than 1/2. Let $n = \pm hk$, where the sign is chosen so that

$$\pm(ak - m) = -|ak - m|.$$
Then with this choice of $n$ we have

$$|e^{2\pi i (b+n\alpha)} - 1| = |e^{2\pi i (b-h|\alpha k - m|)} - 1| \leq 2\pi (b-h|\alpha k - m|) < 2\pi \varepsilon,$$

thanks to the elementary inequality $|e^{2\pi i t} - 1| \leq 2\pi |t|$ (which holds for $0 \leq t \leq 1/2$). This shows that the image of $\gamma_\alpha$ is dense in $S^1 \times S^1$.

The map $\gamma_\alpha$ is not a homeomorphism with its image because its image is dense. Indeed, for $\varepsilon > 0$ consider the interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ and its image $\gamma_\alpha((-\varepsilon, \varepsilon))$. This is not an open neighborhood of $\gamma_\alpha(0)$ in the image $\gamma_\alpha(\mathbb{R})$ with the induced topology, since any small enough neighborhood of $\gamma_\alpha(0)$ intersects infinitely many other “sheets” of $\gamma_\alpha(\mathbb{R})$ and its preimage under $\gamma_\alpha$ has infinitely many other components. Therefore $\gamma_\alpha$ is not an embedding.

3) Determine explicitly the flow $\Theta$ of the vector field

$$X = y \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$
on $M = \mathbb{R}^2$.

**Solution.** Let $\gamma(t) = (f(t), g(t))$ be a smooth curve in $\mathbb{R}^2$. The condition that $\gamma$ be a flow line for $X$ is

$$\gamma'(t) = X_{\gamma(t)},$$
i.e.,

$$f'(t) = g(t), \quad g'(t) = 1.$$

If we set $\gamma(0) = (x_0, y_0)$, then the solution of the ODE for $g$ is $g(t) = y_0 + t$, and so $f'(t) = y_0 + t$, and $f(t) = x_0 + y_0 t + \frac{t^2}{2}$. In particular we see that $X$ is complete, and its flow $\Theta : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\Theta(t, x, y) = \left(x + y t + \frac{t^2}{2}, y + t\right).$$

4) Show that every $n$-manifold $M$ ($n \geq 1$) admits a diffeomorphism $F : M \to M$ which is not the identity.

**Solution.** Let $p$ be any point in $M$ and let $(U, \varphi)$ be a chart centered at $p$ with $\varphi(U) = B_1(0) \subset \mathbb{R}^n$. Let $K = \varphi^{-1}(B_{1/2}(0))$, and let $g$ be a cutoff function which is 1 on $K$ and 0 outside $U$.

If $X = \frac{\partial}{\partial \varphi}$ in this chart (here we use that $n \geq 1$), then $\tilde{X} = gX$ is a well-defined smooth vector field on $M$, compactly supported in $U$ and which
equals $X$ on $K$. Let $\theta_t : M \to M$ be the flow of $\tilde{X}$, which exists for all $t \in \mathbb{R}$ since $\tilde{X}$ is compactly supported hence complete. The point $p$ corresponds to the origin in this chart, and for small positive $t$ we have $\varphi(\theta_t(p)) = te_1$ (where $e_1 = (1, 0, \ldots, 0)$). Therefore $\theta_t(p) \neq p$ for positive small $t$, and hence $\theta_t$ is the desired diffeomorphism.

5) Show that for every $n$-manifold $M$ ($n \geq 1$) the vector space $\mathcal{T}(M)$ of smooth vector fields on $M$ is infinite dimensional.

**Solution.** Let $p$ be any point in $M$ and let $(U, \varphi)$ be a chart centered at $p$ with $\varphi(U) = B_1(0) \subset \mathbb{R}^n$. Let $K = \varphi^{-1}(B_{1/2}(0))$, and let $g$ be a cutoff function which is 1 on $K$ and 0 outside $U$. If $X = \frac{\partial}{\partial x_1}$ in this chart (here we use that $n \geq 1$), and $f$ is a smooth function on $U$, then $X_f := fgX$ is a well-defined smooth vector field on $M$, compactly supported in $U$ and which equals $fX$ on $K$. A linear relation among vector fields of this type implies the same linear relation for the corresponding functions on the interior of $K$. Since the vector space of smooth functions on an open set in $\mathbb{R}^n, n \geq 1$, is infinite dimensional, then so is $\mathcal{T}(M)$. 