1) Let $X$ be a compact complex manifold, $\mathcal{P}$ be the sheaf of (real-valued) pluriharmonic functions on $X$, $C^\infty$ be the sheaf of real-valued smooth functions on $X$, and $Z_R^{1,1}$ be the sheaf of $d$-closed smooth real $(1,1)$ forms on $X$.

(a) Show that we have a short exact sequence of sheaves on $X$

$$0 \to \mathcal{P} \to C^\infty \overset{i\partial\bar{\partial}}{\to} Z_R^{1,1} \to 0$$

(b) Deduce that the Cech cohomology group $H^1(X, \mathcal{P})$ is isomorphic to

$$\{\text{d-closed, real } (1,1)\text{-forms on } X\}$$

$$\sqrt{-1}\partial\bar{\partial}(C^\infty(X, \mathbb{R}))$$

2) Let $X$ be a compact complex manifold. Using the short exact sequence of sheaves on $X$

$$0 \to \mathbb{R} \to \mathcal{O} \overset{\text{Im}}{\to} \mathcal{P} \to 0$$

from Homework 5, prove that $H^1(X, \mathcal{P})$ is isomorphic to $H^{1,1}(X, \mathbb{R})$ if and only if $b_1(X) = 2h^{0,1}(X)$. Recall that we defined in class $H^{1,1}(X, \mathbb{R})$ as the subgroup of $H^2(X, \mathbb{R})$ of (de Rham) classes which have a representative which is a smooth, $d$-closed, real $(1,1)$ form.

3) Let $X$ be a compact complex manifold.

(a) Using the short exact sequence of sheaves on $X$

$$0 \to \mathbb{R} \to \mathcal{O} \overset{\ell}{\to} \mathcal{P} \to 0$$

from Homework 5, define a “refined” first Chern class map

$$\widehat{c}_1 : \text{Pic}(X) \to H^1(X, \mathcal{P}).$$

(b) Show that $\widehat{c}_1(L) = 0$ if and only if $L$ is isomorphic to a holomorphic line bundle whose transition functions (on some trivializing cover) are constant functions with values in the unit circle $\mathbb{R}/\mathbb{Z}$.

4) Let $X$ be a compact complex manifold with $b_1(X) = 2h^{0,1}(X)$ and let $L$ be a holomorphic line bundle on $X$. Using the commutative diagram of sheaves on $X$ (with exact rows)

$$\begin{array}{ccc}
0 & \to & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{C}^* & \overset{\exp}{\to} & \mathbb{C}^* \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{O} \\
\end{array}$$

where $\exp$ is the map $\exp(2\pi i \cdot)$, show that $c_1(L) \in H^2(X, \mathbb{Z})$ is a torsion element if and only if $L$ is isomorphic (as a holomorphic line bundle) to a holomorphic line bundle whose transition functions (on some trivializing cover) are constant functions (with values in $\mathbb{C}^*$, of course).