1) Let \((X, \omega)\) be a compact Kähler manifold of complex dimension \(n > 1\), and let \(u : X \to \mathbb{R}\) be a smooth positive function that satisfies
\[
\Delta u \geq -Au,
\]
where \(A \geq 1\) is a constant, and \(\Delta\) is the Laplacian of the metric \(\omega\). Using the Moser iteration method prove that
\[
\sup_X u \leq CA^n \int_X u \omega^n,
\]
where the constant \(C\) depends only on \((X, \omega)\).

**Solution.** Take \(p > 1\), multiply the equation \(\Delta u \geq -Au\) by \(-u^{p-1}\) and integrate over \(X\) (we will drop the volume form \(dV_g\)).

\[
A \int_X u^p \geq -\int_X u^{p-1} \Delta u = (p-1) \int_X u^{p-2} |\nabla u|^2 = \frac{4(p-1)}{p^2} \int_X |\nabla u^\frac{p}{2}|^2.
\]

For \(p \geq 2\) we can bound \(\frac{4(p-1)}{p^2} \geq \frac{3}{p}\). Now apply the Sobolev inequality to \(u^\frac{p}{2}\) and get
\[
\left( \int_X u^{\frac{2p}{n-1}} \right)^{\frac{n-1}{n}} \leq C \int_X |\nabla u^\frac{p}{2}|^2 + C \int_X u^p,
\]
where \(n\) is the complex dimension of \(X\). Combining these two inequalities we see that
\[
\|u\|_{p\beta} \leq (CAp)^{\frac{1}{\beta}} \|u\|_p,
\]
where \(\|\cdot\|_p\) denotes the \(L^p\) norm, \(\beta = \frac{n}{n-1} > 1\), and \(C\) depends only on the Sobolev constant of \((X, \omega)\). We can then iterate this inequality exactly like we did in class and set \(p = 2\) to get
\[
\|u\|_{\infty} \leq A^\frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{j^2} C\|u\|_2 = CA^\frac{n}{2} \|u\|_2.
\]

Then we can argue that
\[
\|u\|_{\infty} = \sup_X u \leq CA^\frac{n}{2} \|u\|_2 = CA^\frac{n}{2} \left( \int_X u^2 \right)^\frac{1}{2} \leq CA^\frac{n}{2} \|u\|_2 \|u\|_1^\frac{1}{2},
\]
and dividing out gives

$$\sup_X u \leq CA^n \|u\|_1 = CA^n \int_X u,$$

which is what we want.

2) Let $(X, \omega)$ be a compact Kähler manifold of complex dimension $n > 1$, $F : X \to \mathbb{R}$ be a smooth function with $\int_X e^F \omega^n = \int_X \omega^n$ and let $\varphi : X \to \mathbb{R}$ be a smooth function with $\omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0$ solving the complex Monge-Ampère equation

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^F \omega^n.$$

Call $g$ the Hermitian metric defined by $\omega$ and $\tilde{g}$ the one of $\omega + \sqrt{-1} \partial \bar{\partial} \varphi$.

Prove directly that $n + \Delta_g \varphi = \text{tr}_g \tilde{g} \geq C^{-1}$, and $n - \Delta_{\tilde{g}} \varphi = \text{tr}_{\tilde{g}} \tilde{g} \geq C^{-1}$ for some positive constant $C$ that depends only on $\sup_X |F|$ and $n$.

**Solution.** For simplicity fix a point $x \in X$ and pick local coordinates near $x$ so that $g(x) = \text{Id}$ and $\tilde{g}(x)$ is diagonal with eigenvalues $\lambda_1, \ldots, \lambda_n > 0$.

We want to prove first that

$$\text{tr}_{\tilde{g}} \tilde{g}(x) = \sum_{i=1}^n \lambda_i \geq C^{-1}.$$

From the Monge-Ampère equation we have

$$\prod_{i=1}^n \lambda_i = e^{F(x)}.$$

The arithmetic mean-geometric mean inequality gives

$$e^{\frac{F(x)}{n}} = \left( \prod_{i=1}^n \lambda_i \right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^n \lambda_i}{n},$$

and so $\text{tr}_{\tilde{g}} \tilde{g} \geq ne^{-\frac{\int_X F}{n}}$.

We now prove that

$$\text{tr}_{\tilde{g}} \tilde{g}(x) = \sum_{i=1}^n \frac{1}{\lambda_i} \geq C^{-1}.$$
The harmonic mean-geometric mean inequality gives
\[
\left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda_i} \right)^{-1} \leq \left( \prod_{i=1}^{n} \lambda_i \right)^{\frac{1}{n}} = e^{\frac{F(x)}{n}},
\]
and so \( \text{tr} \bar{g} \geq ne^{-\frac{\sup_X F}{n}} \).

3) Assume the same setup as problem 2. Using the Moser iteration method we proved in class that
\[
\|\varphi\|_{L^\infty(X)} \leq C,
\]
where \( C \) depends only on \((X, \omega)\) and \( \sup_X e^F \). Recall that \( \varphi \) is normalized by \( \int_X \varphi \omega^n = 0 \).

Now fix a number \( q > n \). Modify this iteration argument to prove the same \( L^\infty \) estimate for \( \varphi \) with the constant \( C \) depending only on \((X, \omega), q\) and \( \int_X e^{qF} \omega^n \).

**Solution.** Fixing any \( p > 1 \), we proceed exactly like in class and get (we again omit the volume form \( dV_g \))
\[
\int_X |\varphi|^p (1 - e^F) \geq \frac{4(p-1)}{np^2} \int_X |\nabla (|\varphi|^\frac{p-2}{2})|^2,
\]
and we also have
\[
\int_X |\varphi|^p (1 - e^F) = \int_X |\varphi|^p - \int_X |\varphi|^{p-2} e^F \leq \int_X |\varphi|^{p-1} + \int_X |\varphi|^{p-1} e^F.
\]
Using the Sobolev inequality like in class gives us
\[
\|\varphi\|_{p\beta}^p \leq C\|\varphi\|_p^p + Cp\|\varphi\|_{p-1}^{p-1} + Cp \int_X |\varphi|^{p-1} e^F,
\]
where \( \|\cdot\|_p \) denotes the \( L^p \) norm and \( \beta = \frac{n}{n-1} > 1 \). Now use the Hölder inequality
\[
\int_X |\varphi|^{p-1} e^F \leq \left( \int_X |\varphi|^{(p-1)r} \right)^{\frac{1}{r}} \left( \int_X e^{qF} \right)^{\frac{1}{q}} = \|\varphi\|_{(p-1)r}^{p-1} \|e^F\|_q,
\]
where \( \frac{1}{r} + \frac{1}{q} = 1 \). From the Hölder inequality we also have that
\[
\|\varphi\|_{p-1} \leq C\|\varphi\|_{(p-1)r}, \quad \|\varphi\|_p \leq C\|\varphi\|_{pr},
\]
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where $C$ depends only on the volume of $X$ and on $q$. Therefore we get

$$\|\varphi\|_{p\beta} \leq (Cp)^{\frac{1}{p}} (\|\varphi\|_{pr} + \|\varphi\|_{(p-1)r})^2,$$

where $C$ depends on $(X,\omega), q$ and $\|e^F\|_q$. Using Hölder again we have

$$\|\varphi\|_{(p-1)r} \leq \max(1, \|\varphi\|_{pr}),$$

and so

$$\max(1, \|\varphi\|_{p\beta}) \leq (Cp)^{\frac{1}{p}} \max(1, \|\varphi\|_{pr}).$$

Since $q > n$ we have that $r < \frac{n}{n-1} = \beta$, so $p\beta > pr$. Note also that $r < 2$, since $n \geq 2$. Therefore we can iterate this inequality exactly like we did in class (at each iteration the exponent is multiplied by $\frac{\beta}{r} > 1$ instead of $\beta$ as we did in class) and set $p = \frac{2}{r} > 1$ to get

$$\|\varphi\|_\infty \leq C \max(1, \|\varphi\|_2).$$

To bound $\|\varphi\|_2$ we proceed like in class, going back to our first two inequalities and setting $p = 2$ to get

$$\int_X |\nabla \varphi|^2 \leq C \int_X \varphi(1 - e^F) \leq C \int_X |\varphi| + C \|\varphi\|_r \|e^F\|_q \leq C \|\varphi\|_2,$$

using again that $r < 2$. Since $\int_X \varphi \omega^n = 0$, the Poincaré inequality gives

$$\|\varphi\|^2 \leq C \int_X |\nabla \varphi|^2 \leq C \|\varphi\|_2,$$

and so $\|\varphi\|_2 \leq C$, which proves what we want. Note that we cannot lower the condition that $q > n$ with this method, but it is possible to lower it to $q > 1$ with another method (while the result with $q = 1$ is false).