1) Let $\mathcal{D}$ be the distribution on $\mathbb{R}^3$ spanned by the vector fields
\[ X = \frac{\partial}{\partial x} + yz \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}. \]

(a) Write down an integral submanifold for $\mathcal{D}$ passing through the origin.

(b) Is $\mathcal{D}$ integrable?

(c) Can you write down an integral submanifold for $\mathcal{D}$ passing through $(0,0,1)$?

**Solution.** (a) The $xy$-plane $S = \{z = 0\}$ is an integral submanifold for $\mathcal{D}$ passing through the origin. Indeed, at any point $(x,y,0) \in S$ we have $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y}$, which are clearly tangent to $S$ and span its tangent plane.

(b) We compute
\[ [X,Y] = \left( \frac{\partial}{\partial x} + yz \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial y} \right) - \left( \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} + yz \frac{\partial}{\partial z} \right) = -z \frac{\partial}{\partial z}. \]

Since this cannot be written as a linear combination of the form $fX + gY$, with $f, g$ smooth functions, we conclude that $\mathcal{D}$ is not involutive, and so not integrable either.

(c) There is no such submanifold, from the same proof that we saw in class that “$\mathcal{D}$ integrable implies $\mathcal{D}$ involutive”. Briefly, if we have an integral submanifold $S$ then $X, Y$ would be tangent to $S$ and therefore so would be $[X,Y]$. This would imply that for any $p \in S$, $[X,Y]_p = aX_p + bY_p$, for some real numbers $a, b$ (which depend on $p$). But at the point $p = (0,0,1)$ we have $X_p = \frac{\partial}{\partial x}, Y_p = \frac{\partial}{\partial y}, [X,Y]_p = -\frac{\partial}{\partial z}$, and there are no such $a, b$.

Let now
\[ U(n) = \{ A \in \text{Mat}(n, \mathbb{C}) \mid A^*A = \text{Id} \} \]
be the unitary group, where $A^* = \overline{A}^t$, and $\text{Mat}(n, \mathbb{C})$ is the space of $n \times n$ matrices with complex entries, and let
\[ SU(n) = \{ A \in \text{Mat}(n, \mathbb{C}) \mid A^*A = \text{Id} \text{ and } \det A = 1 \} \]
be the special unitary group.
2) Show that $U(n)$ and $SU(n)$ are Lie groups and compute their dimensions.

**Solution.** Clearly $U(n)$ and $SU(n)$ are groups, using matrix multiplication. Let us consider first $U(n)$. Let $\text{Herm}(n)$ be the subspace of $\text{Mat}(n, \mathbb{C})$ of Hermitian matrices (i.e. $A = A^\ast$). A Hermitian matrix is uniquely determined by its entries above the diagonal (which are $n(n-1)/2$ complex numbers) together with its entries on the diagonal (which are $n$ real numbers), hence $\text{Herm}(n)$ is a vector subspace of real dimension $n^2$. Consider the smooth map

$$F : \text{Mat}(n, \mathbb{C}) \to \text{Herm}(n),$$

given by $F(A) = A^\ast A$. We have $dF_A(B) = (F \circ \gamma)'(0)$, where $\gamma : (-\varepsilon, \varepsilon) \to \text{Mat}(n, \mathbb{C}) = \mathbb{R}^{2n^2}$ is any smooth curve with $\gamma(0) = A$ and $\gamma'(0) = B \in T_A \mathbb{R}^{2n^2} = \mathbb{R}^{2n^2}$. We take $\gamma(t) = A + tB$, and compute

$$dF_A(B) = \left. \frac{d}{dt} \right|_{t=0} (A + tB)^\ast (A + tB) = A^\ast B + B^\ast A.$$

Given any $A \in U(n)$ and $C \in \text{Herm}(n)$, we have

$$dF_A \left( \frac{AC}{2} \right) = \frac{1}{2} (A^\ast AC + C^\ast A^\ast A) = C,$$

hence $dF_A$ is surjective. It follows that $U(n)$ is a smooth manifold of real dimension $2n^2 - n^2 = n^2$. Since the group operations are just given by matrix operations, they are smooth, and hence $U(n)$ is a Lie group. Consider now the smooth map

$$\det : U(n) \to S^1.$$ 

This is well-defined since for any $A \in U(n)$ we have

$$1 = \det(A^\ast A) = \det(A^\ast) \det(A) = |\det(A)|^2.$$

It is clearly a Lie group homomorphism, so the rank of its differential is independent of the point. Computing at the identity we have

$$d(\det)_{\text{Id}}(A) = \text{tr}A,$$

for any $A \in u(n)$, i.e. $A$ is skew-Hermitian, and for example the matrix $A \in u(n)$ whose entries are all zero except $a_{11} = \sqrt{-1}$ satisfies $\text{tr}A \neq 0$. Therefore the rank of $d(\det)$ is equal to 1, and so the kernel $SU(n)$ of $\det$ is
a closed Lie subgroup of $U(n)$ of real dimension $n^2 - 1$.

3) Show that $U(1)$ is diffeomorphic to the circle $S^1$.

**Solution.** We have $\text{Mat}(1, \mathbb{C}) = \mathbb{C}$ and $U(1) = \{ z \in \mathbb{C} \mid |z|^2 = 1 \}$. This is the standard description of $S^1$.

4) Show that $SU(2)$ is diffeomorphic to the 3-sphere $S^3$.

**Solution.** Let $A \in SU(2)$, and write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{C}$. Since $\det A = 1$, we have

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The unitary condition is that $A^{-1} = A^*$, and since

$$A^* = \begin{pmatrix} \bar{a} & \bar{c} \\ b & \bar{d} \end{pmatrix},$$

we conclude that $a = \bar{d}$ and $b = -\bar{c}$. Therefore

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

and the condition $\det A = 1$ becomes $|a|^2 + |b|^2 = 1$. Hence we have identified $SU(2)$ with the subset of points $(a, b) \in \mathbb{C}^2$ such that $|a|^2 + |b|^2 = 1$. This is the standard description of $S^3$. 

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