1) Let $\mathcal{D}$ be a $k$-dimensional distribution on a manifold $M$, which is globally spanned by smooth vector fields $X_1, \ldots, X_k$. Let $Y$ be a smooth section of $\mathcal{D}$ over $M$. Show that we can write $Y = \sum_{j=1}^{k} f_j X_j$, where the $f_j$’s are smooth functions on $M$. Deduce that $\mathcal{D}$ is involutive if and only if for any $1 \leq i, j \leq k$ there are smooth functions $f_{ij\ell}$, $1 \leq \ell \leq k$, on $M$ such that $[X_i, X_j] = \sum_{\ell=1}^{k} f_{ij\ell} X_\ell$.

**Solution.** By assumption we have that $(X_1)_p, \ldots, (X_k)_p$ are linearly independent elements of $T_p M$ for all $p \in M$, therefore for any given $p$ we may find unique real numbers $f_j(p)$, $1 \leq j \leq k$, such that

$$Y_p = \sum_{j=1}^{k} f_j(p)(X_j)_p,$$

and we need to prove that the functions $f_j$ so defined are smooth. Since smoothness is a local property, we choose a coordinate chart near $p$ and for each $j$ write

$$X_j = \sum_{\alpha=1}^{n} X_j^\alpha \frac{\partial}{\partial x^\alpha},$$

where $X_j^\alpha$ are smooth functions on this chart. By assumption, the $k \times n$ matrix $\{X_j^\alpha(p)\}$ has a $k \times k$ minor with nonzero determinant (in particular, this remains true in a neighborhood of $p$). Up to permuting the coordinates, we may assume that the $k \times k$ matrix $\{X_j^\alpha(q)\}$ has nonzero determinant, for all $q$ near $p$. In this chart we can also write $Y = Y^\alpha \frac{\partial}{\partial x^\alpha}$, and so for $q$ near $p$ and for $1 \leq \alpha \leq k$ we have

$$Y^\alpha(q) = \sum_{j=1}^{k} f_j(q)X_j^\alpha(q),$$

which gives that $f_j(q)$ equals the inverse matrix of $\{X_j^\alpha(q)\}$ applied to the vector $\{Y^\alpha(q)\}$. Since the entries of this matrix and vector vary smoothly in $q$, it follows that the functions $f_j$ are also smooth.

Now if $\mathcal{D}$ is involutive then $[X_i, X_j]$ belongs to $\mathcal{D}$, and so by what we just proved it can be expressed as $[X_i, X_j] = \sum_{\ell=1}^{k} f_{ij\ell} X_\ell$ for some smooth functions $f_{ij\ell}$. Conversely, assume that $[X_i, X_j] = \sum_{\ell=1}^{k} f_{ij\ell} X_\ell$ and let $X, Y$ be
any two smooth sections of $\mathcal{D}$ over some open subset $U \subset M$. By what we just proved (with $M$ replaced by $U$) we may write

\[ X = \sum_i f_i X_i, \quad Y = \sum_j g_j X_j, \]

and so

\[ [X,Y] = \sum_i \sum_j [f_i X_i, g_j X_j] = \sum_i \sum_j \left( f_i g_j [X_i, X_j] + f_i X_i (g_j) X_j - g_j X_j (f_i) X_i \right) \]

\[ = \sum_i \sum_j \left( \sum_{\ell} f_i g_j f_{i\ell} X_\ell + f_i X_i (g_j) X_j - g_j X_j (f_i) X_i \right), \]

which is a linear combinations of the $X_j$’s and so belongs to $\mathcal{D}$ over $U$, thus proving that $\mathcal{D}$ is involutive.

2) Let $Sp(n, \mathbb{R})$ be the subset of $GL(2n, \mathbb{R})$ of matrices $A$ such that $A^t J A = J$, where

\[ J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \]

for $I_n$ the $n \times n$ identity matrix.

(a) Show that $Sp(n, \mathbb{R})$ is a regular Lie subgroup of $GL(2n, \mathbb{R})$.

(b) Find its dimension.

(c) Determine its Lie algebra $sp(n, \mathbb{R}) \subset gl(n, \mathbb{R})$.

**Solution.** (a) and (b): Note that $J^t = -J$ and $J^t J = I_{2n}$. First of all if $A \in Sp(n, \mathbb{R})$, then $A^t J A = J$ is invertible and hence $A$ is invertible, so $Sp(n, \mathbb{R})$ is indeed a subset of $GL(2n, \mathbb{R})$. It is also a subgroup because if $A^t J A = J$ and $B^t J B = J$, then

\[ (AB)^t J AB = B^t A^t J A B = B^t J B = J. \]

Recall that $\mathfrak{o}(2n)$ equals the space of all $A \in \text{Mat}(2n, \mathbb{R})$ with $A + A^t = 0$. Consider the smooth map

\[ F : GL(2n, \mathbb{R}) \to \mathfrak{o}(2n), \]

given by $F(A) = A^t J A$, so that $Sp(n, \mathbb{R}) = F^{-1}(J)$. We have $dF_A(B) = (F \circ \gamma)'(0)$, where $\gamma : (-\varepsilon, \varepsilon) \to GL(2n, \mathbb{R}) \subset \mathbb{R}^{4n^2}$ is any smooth curve with
\[ \gamma(0) = A \text{ and } \gamma'(0) = B \in T_A\mathbb{R}^{4n^2} = \mathbb{R}^{4n^2}. \] We take \( \gamma(t) = A + tB, \) and compute
\[ dF_A(B) = \left. \frac{d}{dt} \right|_{t=0} (A + tB)^t J(A + tB) = A^t JB + B^t JA. \]

Given any \( A \in Sp(n, \mathbb{R}) \) and \( C \in \mathfrak{o}(2n), \) we have
\[ dF_A \left( -\frac{AJC}{2} \right) = -\frac{1}{2} (A^t JAC + C^t J^t A^t J) = \frac{1}{2} (-C + C^t) = C, \]
hence \( dF_A \) is surjective. Recall that \( \dim \mathfrak{o}(2n) = \frac{2n(2n-1)}{2} \). It follows that \( Sp(n, \mathbb{R}) \) is a smooth submanifold of \( GL(2n, \mathbb{R}) \) of real dimension \( 4n^2 - \frac{2n(2n-1)}{2} = n(2n+1). \) Hence \( Sp(n, \mathbb{R}) \) is a regular Lie subgroup of \( GL(2n, \mathbb{R}). \)

(c) We know that the Lie bracket on \( \mathfrak{sp}(n, \mathbb{R}) \subset \mathfrak{gl}(n, \mathbb{R}) \) is given by \([A, B] = AB - BA, \) so we only need to determine \( \mathfrak{sp}(n, \mathbb{R}) \) as a vector space. By part (a) we know that \( \mathfrak{sp}(n, \mathbb{R}) = \ker dF_{I_{2n}}, \) which means that
\[ \mathfrak{sp}(n, \mathbb{R}) = \{ A \in \mathfrak{gl}(n, \mathbb{R}) \mid A^t J + JA = 0 \}. \]

3) Let \( \pi : E \to M \) be a rank \( r \) vector bundle over a manifold \( M. \) Show that \( M \) is isomorphic to the trivial bundle if and only if it admits \( r \) global smooth sections \( s^1, \ldots, s^r \) with \( s_p^1, \ldots, s_p^r \in E_p \) linearly independent at every point \( p \in M. \) Deduce then that a line bundle is isomorphic to the trivial bundle if and only if it admits a never-vanishing global smooth section.

**Solution.** Fix a covering of \( M \) by open sets \( \{U_\alpha\} \) such that the bundle \( E \) is trivial over each \( U_\alpha. \) Therefore, \( E \) is determined by its transition functions
\[ g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R}). \]
Recall that a global section \( s \) of \( E \) is determined by local vector-valued functions
\[ s_\alpha : U_\alpha \to \mathbb{R}^r, \]
satisfying
\[ s_\alpha = g_{\alpha\beta} \cdot s_\beta, \text{ on } U_\alpha \cap U_\beta, \]
where the dot on the RHS means a matrix applied to a vector. If \( E \) is isomorphic to the trivial bundle, we can take \( g_{\alpha\beta} \equiv \text{Id}, \) and so we can define global sections \( s^1, \ldots, s^r \) by
\[ s^i_\alpha = (0, \ldots, 0, 1, 0, \ldots, 0), \]

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where the component equal to 1 is in the $i$th slot. Clearly, these are linearly independent at each point.

Conversely, let

$$s_\alpha : U_\alpha \to GL(r, \mathbb{R}),$$

be given by the matrix with column vectors $s^1_\alpha, \ldots, s^r_\alpha$, which is invertible because $s^1, \ldots, s^r$ which are linearly independent at every point. Then we have

$$s_\alpha = g_{\alpha \beta} s_\beta, \quad \text{on } U_\alpha \cap U_\beta,$$

where multiplication on the RHS is matrix multiplication. Since $s_\alpha$ is invertible, we define

$$\sigma_\alpha = (s_\alpha)^{-1} : U_\alpha \to GL(r, \mathbb{R}).$$

Then we have

$$\text{Id} \equiv \sigma_\alpha g_{\alpha \beta} \sigma_\beta^{-1},$$

i.e. $\sigma_\alpha g_{\alpha \beta} \sigma_\beta^{-1}$ are the transition functions of a bundle isomorphic to $E$, which is trivial as required.

Lastly, in the case of a line bundle just note that a global section $s$ is linearly independent at each point if and only if it is never vanishing.

4) Let $G$ be a Lie group. Show that its tangent bundle $TG$ is isomorphic to the trivial bundle.

**Solution.** Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g} = T_eG$. Denote by $\tilde{X}_1, \ldots, \tilde{X}_n$ their extension as left-invariant vector fields on $G$. Then for each $g \in G$, $(\tilde{X}_1)_g, \ldots, (\tilde{X}_n)_g$ form a basis of $T_gG$, because they are the image of the basis $X_1, \ldots, X_n$ under the linear isomorphism $d(L_g)_e$.

The result then follows from exercise 3).

5) As in class, cover $\mathbb{RP}^n$ by the canonical covering $\{U_\alpha\}_{0 \leq \alpha \leq n}$, where

$$U_\alpha = \{[x_0 : \ldots : x_n] \in \mathbb{RP}^n \mid x_\alpha \neq 0\},$$

and for each $d \in \mathbb{Z}$ define a line bundle $E_d$ by the transition functions

$$g_{\alpha \beta} : U_\alpha \cap U_\beta \to \mathbb{R}^*, \quad g_{\alpha \beta} = \left(\frac{x_\beta}{x_\alpha}\right)^d.$$

(a) Show that $E_d \cong E_1 \otimes \cdots \otimes E_1$ ($d$ times) for $d > 0$, and $E_d \cong (E_1 \otimes \cdots \otimes E_1)^*$ for $d < 0$. 

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(b) Show that $E_d$ is isomorphic to the trivial bundle if $d$ is even.

(c) Show that $E_d$ is isomorphic to $E_1$ if $d$ is odd.

(d) Show that $E_1$ is not isomorphic to the trivial bundle.

Solution. (a) Recall first the canonical covering $\{U_\alpha\}_{0 \leq \alpha \leq n}$ of $\mathbb{RP}^n$, where

$$U_\alpha = \{[x_0 : \cdots : x_n] \in \mathbb{RP}^n \mid x_\alpha \neq 0\},$$

and chart maps are

$$\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^n,$$

$$\varphi_\alpha([x_0 : \cdots : x_n]) = \left(\frac{x_0}{x_\alpha}, \cdots, \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, \cdots, \frac{x_n}{x_\alpha}\right).$$

On this covering the transition functions of $E_d$ are given by

$$g_{\alpha\beta} = \left(\frac{x_\beta}{x_\alpha}\right)^d.$$

In general, given a line bundle $L$ with transition functions $g_{\alpha\beta}$, the line bundle $L^d$, $d \in \mathbb{Z}$ has transition functions $g_{\alpha\beta}^d$. Hence (a) follows.

(b) Thanks to exercise 3) it is enough to construct a global section $s : \mathbb{RP}^n \rightarrow E_d$ which is never vanishing. Thanks to part (a) we may assume that $d > 0$. Then define

$$s_\alpha : U_\alpha \rightarrow \mathbb{R}^*,$$

by

$$s_\alpha([x_0 : \cdots : x_n]) = \frac{x_0^d + \cdots + x_n^d}{x_\alpha^d}.$$}

This is indeed a smooth never-vanishing function on $U_\alpha$, because $d$ is even and so the numerator and denominator are strictly positive. On $U_\alpha \cap U_\beta$ we have

$$s_\alpha = \frac{x_0^d + \cdots + x_n^d}{x_\alpha^d} = \left(\frac{x_\beta}{x_\alpha}\right)^d \frac{x_0^d + \cdots + x_n^d}{x_\beta^d} = g_{\alpha\beta} s_\beta,$$

and so they define a smooth global never vanishing section $s$ of $E_d$.

(c) Write $E_d = E_{d-1} \otimes E_1$, and use (b).

(d) If $E_1$ was trivial, it would have a never-vanishing section $s$, locally given by

$$s_\alpha : U_\alpha \rightarrow \mathbb{R}^*,$$

with

$$s_\alpha = \left(\frac{x_\beta}{x_\alpha}\right) s_\beta, \text{ on } U_\alpha \cap U_\beta.$$
In particular,

\[
\text{sgn}(s_\alpha) = \frac{s_\alpha}{|s_\alpha|} \in \{\pm 1\},
\]

is a well-defined constant on \(U_\alpha\), since \(U_\alpha\) is connected. Hence, on \(U_\alpha \cap U_\beta\) we have

\[
\text{sgn}(s_\alpha) = \text{sgn}\left(\frac{x_\beta}{x_\alpha}\right) \text{sgn}(s_\beta).
\]

However, \(U_\alpha \cap U_\beta \cong \{x \in \mathbb{R}^n \mid x_1 \neq 0\}\) has two connected components, and the function \(\text{sgn}\left(\frac{x_\alpha}{x_\beta}\right)\) has a different value on each component, while \(\text{sgn}(s_\alpha)\) and \(\text{sgn}(s_\beta)\) have the same value on both components, a contradiction.