1) Let $M_1, M_2$ be two compact $n$-manifolds and for each $j = 1, 2$ let $B_j \subset M_j$ be open subsets with $\psi_j : B_j \to B_1(0) \subset \mathbb{R}^n$ diffeomorphisms. Let $B_j' = \psi_j^{-1}(B_{1/2}(0))$ and $M_j' = M_j \setminus B_j'$. Let the connected sum of $M_1$ and $M_2$ be

$$M_1 \sharp M_2 = (M_1' \cup M_2') / \sim,$$

where the equivalence relation $\sim$ identifies $x_1 \in B_1 \setminus B_1'$ with $x_2 \in B_2 \setminus B_2'$ iff $\psi_1(x_1) = \psi_2(x_2)$. We have that $M_1 \sharp M_2$ is also a compact $n$-manifold. Show that the Euler characteristic of $M_1 \sharp M_2$ is given by

$$\chi(M_1 \sharp M_2) = \chi(M_1) + \chi(M_2) - \chi(S^{n-1}).$$

**Solution:** First, let us compute the Euler characteristic of $M_1'$. For this, cover $M_1$ by the open sets $M_1', B_1$, and apply Mayer-Vietoris to get the exact sequence

$$\cdots \to H^k(M_1) \to H^k(M_1') \oplus H^k(B_1) \to H^k(M_1' \cap B_1) \to \cdots$$

Note that $B_1$ is contractible, so $\chi(B_1) = 1$, and $M_1' \cap B_1$ is homotopy equivalent to $S^{n-1}$, so $\chi(M_1' \cap B_1) = 1 + (-1)^{n-1}$. Since all the cohomology groups in the exact sequence are finite dimensional, it follows (from an elementary algebraic result) that

$$\chi(M_1) = \chi(M_1') + \chi(B_1) - \chi(M_1' \cap B_1) = \chi(M_1') + 1 - \chi(S^{n-1}).$$

Similarly for $M_2'$. Then we cover $M_1 \sharp M_2$ by the open sets which are (the images of) $M_1', M_2'$, and apply Mayer-Vietoris to get the exact sequence

$$\cdots \to H^k(M) \to H^k(M_1') \oplus H^k(M_2') \to H^k(M_1' \cap M_2') \to \cdots$$

Now $M_1' \cap M_2'$ is homotopy equivalent to $S^{n-1}$, and so

$$\chi(M) = \chi(M_1') + \chi(M_2') - \chi(S^{n-1}) = \chi(M_1) + \chi(M_2) + \chi(S^{n-1}) - 2,$$

and clearly

$$\chi(S^{n-1}) - 2 = 1 + (-1)^{n-1} - 2 = -(1 + (-1)^n) = -\chi(S^n),$$

and we are done.
2) Show that $\mathbb{CP}^2 \times S^2$ is not homotopy equivalent to $\mathbb{CP}^4 \vee S^2$.

**Solution:** By Künneth

$$H^8(\mathbb{CP}^2 \times S^2) = 0,$$

but

$$H^8(\mathbb{CP}^4 \vee S^2) \cong H^8(\mathbb{CP}^4) \oplus H^8(S^2) \cong \mathbb{R},$$

and so these two spaces are not homotopy equivalent.

3) Let $X = S^2/\sim$, where $\sim$ is the equivalence relation which identifies any point $x$ on the equator $S^1 \subset S^2$ with $-x$.

(a) Show that $X$ has the structure of a CW complex.

(b) Write down the cellular chain complex for the homology of $X$.

(c) Use it to compute $H_k(X; \mathbb{Z})$ and $\tilde{H}_k(X; \mathbb{Z})$, for all $k$.

**Solution:** (a) We let $X^0 = \{\text{pt}\}$, we attach a single 1-cell $e^1$ to $X^0$, via the obvious map (so $X^1 \cong S^1$), and we attach two 2-cells $e^2_1, e^2_2$ to $X^1$, by wrapping $S^1 = \partial e^2_1$ to $X^1 = S^1$ twice clockwise, and wrapping $S^1 = \partial e^2_2$ to $X^1 = S^1$ twice counterclockwise. The resulting space is homeomorphic to $X$.

(b) From the description in (a), we have that the cellular chain complex is

$$0 \to \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0,$$

and clearly $d_1(e^1) = 0$, so $d_1$ is the zero map. From the cellular boundary formula, $d_2(e^2_1) = 2e^1, d_2(e^2_2) = -2e^1$. Therefore $d_2(xe^2_1 + ye^2_2) = 2(x-y)e^1$, for any $x, y \in \mathbb{Z}$.

(c) Thanks to (b), we have $H_0(X; \mathbb{Z}) \cong \mathbb{Z}$. The kernel of $d_2$ is isomorphic to $\mathbb{Z}$, given by multiples of $e^2_1 - e^2_2$, and so $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$. Lastly, the image of $d_2$ is the subgroup generated by $2e^1$, so $H_1(X; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

4) Let $\mathbb{CP}^1$ be the Riemann sphere and $\mathcal{O}$ the sheaf of holomorphic functions on $\mathbb{CP}^1$. Consider the standard covering $\mathcal{U} = \{U, V\}$, where in homogeneous coordinates $[z_0 : z_1]$ we have set $U = \{z_0 \neq 0\}, V = \{z_1 \neq 0\}$, with local holomorphic coordinates $u = \frac{z_1}{z_0}$ on $U \cong \mathbb{C}$ and $v = \frac{z_0}{z_1}$ on $V \cong \mathbb{C}$. On the overlap $U \cap V \cong \mathbb{C}^*$ these two coordinates are related by $v = \frac{1}{u}$.
Using the definition, compute the Čech cohomology groups $H^0(U, \mathcal{O})$ and $H^1(U, \mathcal{O})$.

(In fact, these groups coincide with $H^0(\mathbb{CP}^1, \mathcal{O})$ and $H^1(\mathbb{CP}^1, \mathcal{O})$, but this is a bit harder to prove).

Solution: By definition we have

$$C^0(U, \mathcal{O}) = \{ (f, g) \mid f \in \mathcal{O}(U), g \in \mathcal{O}(V) \},$$

$$C^1(U, \mathcal{O}) = \{ h \mid h \in \mathcal{O}(U \cap V) \}.$$

Given $(f, g) \in C^0(U, \mathcal{O})$, we expand in power series

$$f = \sum_{n=0}^{\infty} a_n u^n,$$

on $U \cong \mathbb{C}$ and

$$g = \sum_{n=0}^{\infty} b_n v^n,$$

on $V \cong \mathbb{C}$. On the overlap $U \cap V \cong \mathbb{C}^*$ we can therefore write

$$g = \sum_{n=0}^{\infty} b_n u^{-n},$$

and by definition we have that the Čech differential $\delta$ applied to $(f, g) \in C^0(U, \mathcal{O})$ equals

$$\delta((f, g)) = (-f + g)|_{U \cap V} = \sum_{n=0}^{\infty} a_n u^n + \sum_{n=0}^{\infty} b_n u^{-n},$$

and so $\delta((f, g)) = 0$ iff $a_n = b_n = 0$ for all $n > 0$ and $a_0 = b_0$, which can be an arbitrary complex number. Therefore, we have identified

$$H^0(U, \mathcal{O}) = \ker \delta \cap C^0(U, \mathcal{O}) \cong \mathbb{C}.$$

On the other hand every $h \in \mathcal{O}(U \cap V)$ can be expanded in Laurent series

$$h = \sum_{n=-\infty}^{\infty} a_n u^n = \sum_{n=-\infty}^{\infty} a_n v^{-n},$$

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on $U \cap V \cong \mathbb{C}^*$. If we set

$$f = -\sum_{n=0}^{\infty} a_n u^n,$$

and

$$g = \sum_{n=0}^{\infty} a_{-n} v^n,$$

then by construction we have

$$h = \delta((f, g)),$$

and we have shown that

$$H^1(U, \mathcal{O}) = \frac{\mathcal{O}(U \cap V)}{\text{Im} \delta \cap C^1(U, \mathcal{O})} = 0.$$