1) Let \((X, \omega)\) be a compact Kähler manifold of complex dimension \(n\) with \(\text{Ric}(\omega) \leq -\varepsilon \omega\) for some \(\varepsilon > 0\). Let \(\tilde{\omega}\) be another Kähler metric on \(X\) with scalar curvature \(\tilde{R} \geq -A\) for some \(A > 0\).

Applying the minimum principle to the quantity \(\log \frac{\tilde{\omega}^n}{\omega^n}\), prove that \(\tilde{\omega}^n \geq C^{-1} \omega^n\), for some constant \(C\) that depends only on \(\varepsilon, A, n\).

**Solution.** Calculate
\[
\Delta_{\tilde{g}} \log \frac{\tilde{\omega}^n}{\omega^n} = -\tilde{R} + \text{tr}_{\tilde{g}} \text{Ric}(\omega) \leq A - \varepsilon \text{tr}_{\tilde{g}} g.
\]

At a point \(x \in X\) where \(\log \frac{\tilde{\omega}^n}{\omega^n}\) achieves its minimum we have
\[
0 \leq \Delta_{\tilde{g}} \log \frac{\tilde{\omega}^n}{\omega^n}(x) \leq A - \varepsilon \text{tr}_{\tilde{g}} g(x),
\]
and so \(\text{tr}_{\tilde{g}} g(x) \leq \frac{A}{\varepsilon}\). Using the harmonic mean-geometric mean inequality as in homework 6, question 2 (with the notation from there) we get
\[
\frac{\tilde{\omega}^n}{\omega^n}(x) = \prod_{i=1}^{n} \lambda_i \geq \left( \frac{\sum_{i=1}^{n} 1}{n} \right)^{-n} = \left( \frac{\text{tr}_{\tilde{g}} g(x)}{n} \right)^{-n} \geq (\frac{\varepsilon n}{A})^{-n} = C^{-1}.
\]

In particular, \(\log \frac{\tilde{\omega}^n}{\omega^n}(x) \geq -\log C\), and since \(x\) is the minimum, this is true everywhere, which gives \(\tilde{\omega}^n \geq C^{-1} \omega^n\), as required.

2) Let \((X, \omega)\) be a compact Kähler manifold of complex dimension \(n \geq 2\), and denote by \(R\) the scalar curvature of \(\omega\). Prove that
\[
\int_{X} R^2 \omega^n \geq n^2 \int_{X} c_1(X)^2 \wedge \omega^{n-2},
\]
with equality if and only if \(\omega\) is Kähler-Einstein (so \(\text{Ric}(\omega) = \lambda \omega\) for some \(\lambda \in \mathbb{R}\)).

**Solution.** Consider the tensorial length squared of \(\text{Ric}(\omega) - \frac{R}{n} \omega\)
\[
\left| \text{Ric} - \frac{R}{n} \omega \right|^2 = |\text{Ric}|^2 - \frac{R^2}{n}.
\]
Integrate this over $X$ with respect to $\omega^n$ and get
\[ \int_X |Ric|^2 \omega^n - \frac{1}{n} \int_X R^2 \omega^n = \int_X \left| \frac{R}{n} \omega \right|^2 \omega^n \geq 0, \]
with equality if and only if $\omega$ is Kähler-Einstein. We now claim that
\[ \int_X |Ric|^2 \omega^n = \int_X R^2 \omega^n - n(n-1) \int_X c_1^2(X) \wedge \omega^{n-2}. \]
This is because $c_1(X)$ is represented by the Ricci form $\text{Ric}(\omega) = \sqrt{-1} R_{i\overline{j}} dz^i \wedge d\overline{z}^j$ and $c_1^2(X)$ is represented by
\[ \text{Ric}(\omega)^2 = (\sqrt{-1})^2 R_{i\overline{j}} R_{k\overline{l}} dz^i \wedge d\overline{z}^j \wedge dz^k \wedge d\overline{z}^l. \]
If we pick coordinates at a point so what $\omega = \sqrt{-1} \sum_i dz^i \wedge d\overline{z}^i$ is diagonal, then
\[ n(n-1)\text{Ric}(\omega)^2 \wedge \omega^{n-2} = \sum_{i\neq j} (R_{i\overline{i}} R_{j\overline{j}} - R_{i\overline{j}} R_{j\overline{i}}) \omega^n \]
\[ = \sum_{i,j} (R_{i\overline{i}} R_{j\overline{j}} - R_{i\overline{j}} R_{j\overline{i}}) \omega^n \]
\[ = (R^2 - |\text{Ric}|^2) \omega^n, \]
and integrating proves our claim. Therefore we get that
\[ \left( 1 - \frac{1}{n} \right) \int_X R^2 \omega^n \geq n(n-1) \int_X c_1^2(X) \wedge \omega^{n-2}, \]
with equality if and only if $\omega$ is Kähler-Einstein. This is exactly what we need to prove.

3) Let $X$ be a K3 surface, that is a compact Kähler manifold of complex dimension 2 which is simply connected and with $c_1(X) = 0$. Find the exact universal value $\beta$ of the $L^2$ norm of the Riemann curvature tensor of $\omega$, any Ricci-flat Kähler metric on $X$
\[ \int_X |\text{Rm}|^2 \frac{\omega^2}{2!} = \beta. \]
(You can use the fact that every K3 surface has Betti number $b_2(X) = 22$). Then prove that for any other Kähler metric $\omega'$ on $X$ we have
\[ \int_X |\text{Rm}'|^2 \frac{\omega'^2}{2!} \geq \beta, \]
with equality if and only if $\omega'$ is Ricci-flat. (Note that a similar statement on a complex torus is obvious, with $\beta = 0$).

**Solution.** We calculated in class that for any Kähler metric $\omega$ on any compact Kähler manifold of complex dimension $n$ we have

$$\int_X |\text{Rm}|^2 \omega^n - \int_X |\text{Ric}|^2 \omega^n = n(n-1) \int_X (2c_2(X) - c_1^2(X)) \wedge \omega^{n-2}.$$ 

For your convenience, here is the proof: let $\Omega^i_j = \sqrt{-1} R^j_{ik\ell} dz^k \wedge dz^\ell$ denote the curvature forms, then the form

$$\text{tr}(\Omega \wedge \Omega) = \sum_{k,i} \Omega^k_i \wedge \Omega^i_k = (\sqrt{-1})^2 R^k_{ipq} R^i_{kr\tau} dz^p \wedge dz^q \wedge dz^r \wedge dz^\tau$$

represents $c_1^2(X) - 2c_2(X)$. If we pick coordinates at a point so what $\omega = \sqrt{-1} \sum_i dz^i \wedge dz^i$ is diagonal, then we can compute that

$$n(n-1)\text{tr}(\Omega \wedge \Omega) \wedge \omega^{n-2} = \sum_{p \neq r} (R^k_{ipq} R^i_{kr\tau} - R^k_{ip\tau} R^i_{kr\tau}) \omega^n$$

$$= \sum_{p \neq r} (R^k_{ipq} R^i_{kr\tau} - R^k_{ip\tau} R^i_{kr\tau}) \omega^n$$

$$= (|\text{Ric}|^2 - |\text{Rm}|^2) \omega^n,$$

and integrating gives what we claimed.

If $n = 2$ and $c_1(X) = 0$ this reduces to

$$\int_X |\text{Rm}|^2 \omega^2 \frac{2!}{2!} - \int_X |\text{Ric}|^2 \omega^2 \frac{2!}{2!} = 2 \int_X c_2(X).$$

The Chern-Gauss-Bonnet theorem says that

$$\int_X c_2(X) = 4\pi^2 \chi(X).$$

On a $K3$ surface we have $b_1(X) = b_3(X) = 0$ by Poincaré duality and $b_2(X) = 22$, so $\chi(X) = 24$. Therefore, for any Kähler metric on any $K3$ surface we have

$$\int_X |\text{Rm}|^2 \omega^2 \frac{2!}{2!} = \int_X |\text{Ric}|^2 \omega^2 \frac{2!}{2!} + 192\pi^2 \geq 192\pi^2 = \beta,$$

with equality if and only if $\omega$ is Ricci-flat.

3