1) Let $E \to M$ be a vector bundle, and $F \to M$ be the trivial rank $r$ vector bundle. Show that $E \otimes F \cong E \oplus \cdots \oplus E \ (r \text{ times}).$

**Solution.** Cover $M$ by an open covering $\{U_\alpha\}$ so that $E$ is trivial over each $U_\alpha$, and let $g_{\alpha\beta}$ be its transition functions. The trivial bundle $F$ has transition functions $h_{\alpha\beta} \equiv \text{Id}_r$, the $r \times r$ identity matrix, on the same covering. We know that the transition functions of $E \otimes F$ are the Kronecker product $g_{\alpha\beta} \otimes h_{\alpha\beta}$. Since $h_{\alpha\beta} \equiv \text{Id}_r$, this Kronecker product is the block matrix

$$
\begin{pmatrix}
g_{\alpha\beta} & 0 & \cdots & 0 \\
0 & g_{\alpha\beta} & \cdots & 0 \\
& \cdots & \ddots & \vdots \\
0 & 0 & \cdots & g_{\alpha\beta}
\end{pmatrix},
$$

with $r$ blocks. These are exactly the transition functions of the direct sum bundle $E \oplus \cdots \oplus E \ (r \text{ times}).$

2) Let $M$ be a manifold. Show that the set of isomorphism classes of line bundles over $M$ is an Abelian group, with group multiplication being tensor product.

**Solution.** Let $P$ be this set. We declare the identity element $e$ to be the isomorphism class of the trivial bundle. If $g, h \in P$ are two isomorphism classes, pick representatives $L_1 \to M$ of $g$ and $L_2 \to M$ of $h$. Then declare the isomorphism class of $L_1 \otimes L_2$ to be the product $gh$. This is well-defined, since changing representatives $L_1, L_2$ gives rise to an isomorphic tensor product. Lastly, if $g \in P$, pick a representative $L$ of $g$, and declare $g^{-1}$ to be the isomorphism class of the dual bundle $L^*$, which is again well-defined.

First, let us check that $gg^{-1} = g^{-1}g = e$. If $L$ represents $g$ and has transition functions $g_{\alpha\beta}$ on some trivializing covering of $M$, then $L^*$ has transition functions $g_{\alpha\beta}^{-1}$, and so $L \otimes L^*$ has transition functions $g_{\alpha\beta}g_{\alpha\beta}^{-1} = 1$, and hence it is trivial. On the other hand, $L^* \otimes L$ has transition functions $g_{\alpha\beta}^{-1}g_{\alpha\beta} = 1$, and hence it is trivial too.

Second, let us check that $ge = eg = g$. If $L$ represents $g$ and has transition functions $g_{\alpha\beta}$ on some trivializing covering of $M$, and $E$ is the trivial line bundle, with transition functions identically equal to 1, then $L \otimes E$ and
$E \otimes L$ have both transition functions $g_{\alpha\beta} \cdot 1 = g_{\alpha\beta}$, and so are isomorphic to $L$.

Third, let us check that $g(hk) = (gh)k$. If $L$ represents $g$ and has transition functions $g_{\alpha\beta}$, $N$ represents $h$ and has transition functions $h_{\alpha\beta}$ and $Q$ represents $k$ and has transition functions $k_{\alpha\beta}$, then $L \otimes (N \otimes Q)$ has transition functions $g_{\alpha\beta}h_{\alpha\beta}k_{\alpha\beta}$, and so does $(L \otimes N) \otimes Q$.

Lastly, let us check that $gh = hg$. If $L$ represents $g$ and has transition functions $g_{\alpha\beta}$ and $N$ represents $h$ and has transition functions $h_{\alpha\beta}$, then $L \otimes N$ represents $gh$ and has transition functions $g_{\alpha\beta}h_{\alpha\beta}$, and so does $N \otimes L$.

We conclude that $P$ is an Abelian group.

3) Let $\pi : E_1 \to \mathbb{R}P^1$ be the line bundle over $\mathbb{R}P^1$ that we considered in homework 7. Show that the total space of $E_1$, which is a 2-dimensional manifold, is not orientable.

**Solution.** Let $[x_0 : x_1]$ be the usual homogeneous coordinates on $\mathbb{R}P^1$, and cover it by the two charts $U_\alpha = \{x_\alpha \neq 0\} \cong \mathbb{R}$, $\alpha = 0, 1$. The chart maps are $\varphi_0([x_0 : x_1]) = x_1/x_0$ and $\varphi_1([x_0 : x_1]) = x_0/x_1$, and the transition functions are $\varphi_0 \circ \varphi_1^{-1}(x) = 1/x = \varphi_1 \circ \varphi_0^{-1}(x)$, which are functions from $\mathbb{R}^* \cong U_0 \cap U_1$ to itself.

The transition functions of the bundle $E_1$ are $g_{\alpha\beta} = x_\beta/x_\alpha$, $\alpha, \beta = 0, 1$. The manifold $E_1$ is covered by two charts, $\psi_\alpha : \pi^{-1}(U_\alpha) \cong \varphi_\alpha(U_\alpha) \times \mathbb{R}$, $\alpha = 0, 1$.

As computed in class, we have

$$\psi_\alpha \circ \psi_1^{-1}(x, v) = (\varphi_\alpha \circ \varphi_1^{-1}(x), g_{\alpha\beta}(x)(v)).$$

Therefore, the transition functions of $E_1$ as a manifold are

$$\psi_0 \circ \psi_1^{-1} : \mathbb{R}^* \times \mathbb{R} \to \mathbb{R}^* \times \mathbb{R}, \quad (x, v) \mapsto \left(\frac{1}{x}, \frac{v}{x} \right),$$

and $\psi_1 \circ \psi_0^{-1}$ which is given by the same formula. For notational ease, call this map $F$. Then the Jacobian of $F$ is

$$dF = \begin{pmatrix} -\frac{1}{x^2} & -\frac{v}{x^2} \\ 0 & \frac{1}{x} \end{pmatrix},$$

and its determinant is $-\frac{1}{x^2}$, which is positive on one of the components of $\mathbb{R}^*$ and negative on the other.

If $E_1$ was orientable, there would be a volume form $\nu \in \Lambda^2 E_1$. On $U_0 \times \mathbb{R} \cong \mathbb{R}^2$, with coordinates $(x, v)$, we can write $\nu = f dx \wedge dv$, with $f$ smooth.
and never zero (hence of constant sign, since $\mathbb{R}^2$ is connected), while on $U_1 \times \mathbb{R} \cong \mathbb{R}^2$, with coordinates $(y, w)$, we can write $\nu = g dy \wedge dw$, with $g$ smooth and of constant sign. On the overlap, we have

$$dy \wedge dw = \det(dF) dx \wedge dv,$$

and so

$$fdx \wedge dv = gdy \wedge dw = g \det(dF) dx \wedge dv,$$

which implies

$$f = g \det(dF),$$

and since $\det(dF)$ changes sign, this gives a contradiction.

4) Let $M$ be an $n$-manifold, and $X_1, \ldots, X_n$ be smooth vector fields on an open set $U \subset M$ which are linearly independent at each point. Let $\alpha_1, \ldots, \alpha_n$ be the smooth 1-forms on $U$ which at each point are the dual basis of these vector fields. On $U$ we can write

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k,$$

for smooth functions $c_{ij}^k$ on $U$, for all $1 \leq i, j \leq n$ (recall homework 7). Show that

$$d\alpha_k = - \sum_{1 \leq i < j \leq n} c_{ij}^k \alpha_i \wedge \alpha_j,$$

on $U$, for all $1 \leq k \leq n$.

**Solution.** A basis of 2-forms on $U$ is given by $\{\alpha_i \wedge \alpha_j\}, 1 \leq i < j \leq n$. Write

$$d\alpha_k = \sum_{1 \leq i < j \leq n} d_{ij}^k \alpha_i \wedge \alpha_j,$$

for smooth functions $d_{ij}^k$ on $U$, for all $1 \leq k \leq n$, and $1 \leq i < j \leq n$. For any $1 \leq k \leq n$ and $1 \leq a < b \leq n$ we have

$$d\alpha_k(X_a, X_b) = \sum_{1 \leq i < j \leq n} d_{ij}^k (\alpha_i \wedge \alpha_j)(X_a, X_b)$$

$$= \sum_{1 \leq i < j \leq n} d_{ij}^k (\alpha_i(X_a)\alpha_j(X_b) - \alpha_i(X_b)\alpha_j(X_a))$$

$$= \sum_{1 \leq i < j \leq n} d_{ij}^k (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}) = d_{ab}^k.$$
since $a < b$ implies $\sum_{1 \leq i < j \leq n} \delta_{ib}\delta_{ja} = 0$. On the other hand, we also have

$$d\alpha_k(X_a, X_b) = X_a(\alpha_k(X_b)) - X_b(\alpha_k(X_a)) - \alpha_k([X_a, X_b])$$

$$= X_a(\delta_{kb}) - X_b(\delta_{ka}) - \sum_{\ell=1}^n c_{ab}^{\ell} \alpha_k(X_\ell)$$

$$= -c_{ab}^k,$$

hence $d_{ij}^k = -c_{ij}^k$ for $1 \leq k \leq n$, $1 \leq i < j \leq n$. 

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