Solve all the problems below. Each problem is worth 10 points.

1) Let \( X \) be a path connected and locally path connected space with \( \pi_1(X,b_0) \) finite. Show that every continuous map \( f : X \to \mathbb{S}^1 \) is nullhomotopic.

**Solution.** Let \( p : \mathbb{R} \to \mathbb{S}^1 \) be the universal cover. Now \( f_* : \pi_1(X,b_0) \to \pi_1(\mathbb{S}^1,f(b_0)) \) is a homomorphism from a finite group to \( \mathbb{Z} \), hence it is trivial (since \( \mathbb{Z} \) has no nontrivial elements of finite order). Therefore \( f_* \pi_1(X,b_0) \subset p_* \pi_1(\mathbb{R},e_0) \) and so by the lifting property \( f \) admits a lift to \( \tilde{f} : X \to \mathbb{R} \).

Since \( \mathbb{R} \) is contractible, there is a homotopy \( F \) from \( \tilde{f} \) to a constant map, and so \( p \circ F \) is a homotopy from \( f \) to a constant map.

2) Let \( X = \mathbb{S}^1 \times \mathbb{S}^1 \) be the 2-torus, which we embed in \( \mathbb{C}^2 \) as usual by

\[
\mathbb{S}^1 \times \mathbb{S}^1 = \{(z,w) \in \mathbb{C}^2 \mid |z| = 1 = |w|\}.
\]

Let \( f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1 \) be the continuous map given by

\[
f(z,w) = (z^a w^b, z^c w^d),
\]

for some \( a, b, c, d \in \mathbb{Z} \). Compute the induced homomorphism \( f_* \) from \( \pi_1(\mathbb{S}^1 \times \mathbb{S}^1,b_0) \cong \mathbb{Z}^2 \) to itself.

**Solution.** The inverse of the isomorphism \( \pi_1(\mathbb{S}^1 \times \mathbb{S}^1,b_0) \cong \mathbb{Z}^2 \) is given by \( \mathbb{Z}^2 \ni (m,n) \mapsto \gamma(t) = (e^{2\pi i mt}, e^{2\pi i nt}) \). We have

\[
f \circ \gamma(t) = (e^{2\pi i (am+bn)t}, e^{2\pi i (cm+dn)t}),
\]

and so

\[
f_*(m,n) = (am + bn, cm + dn),
\]

or in matrix form

\[f_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

3) Let \( X \) be the topological space obtained from the (filled) triangle by identifying all of its three sides as shown:

![Diagram](image)

Calculate its fundamental group.

**Solution.** Let \( U \) be the open set in \( X \) which is the image of almost all the interior of the triangle, and \( V \) the open set which is the image of a small neighborhood of the boundary triangle, so that \( X = U \cup V \), \( U \) and \( V \) are path connected, \( U \) is contractible, \( V \) is deformation equivalent to the image of the boundary triangle, which is homeomorphic to \( \mathbb{S}^1 \), and \( U \cap V \) is path connected and deformation equivalent to \( \mathbb{S}^1 \). Let \( \gamma \) be a loop in \( U \cap V \) which is homotopic to the loop which goes along the boundary triangle once clockwise. Then \([\gamma]\) generates \( \pi_1(V,z_0) \) and \( \pi_1(U \cap V,z_0) \) and if \( i : \pi_1(U \cap V,z_0) \to \pi_1(V,z_0) \), \( j : \pi_1(U \cap V,z_0) \to \pi_1(U,z_0) \) are the homomorphism induced by the inclusions then we have that \( j \) is the trivial homomorphism while \( i([\gamma]) \) is the homotopy class of the path \( \gamma \) in \( V \), which is the class of the image of the loop which goes along the boundary triangle.
once clockwise, and this becomes the class of $[\gamma] * [\gamma] * [\gamma]$ in $\pi_1(V, z_0)$. In other words, we have $i([\gamma])j([\gamma])^{-1} = 3[\gamma]$. By Seifert-van Kampen we have

$$\pi_1(X, b_0) \cong (\pi_1(U, b_0) * \pi_1(V, b_0))/N,$$

where $N$ is the normal subgroup generated by $3[\gamma]$, and so

$$\pi_1(X, b_0) \cong \mathbb{Z}/3\mathbb{Z}.$$

4) Let $X$ be the same space as in problem 3.

(a) Is there a covering space $p : E \to X$ with $E$ path connected and locally path connected, such that $p$ has 2 sheets (i.e. $p^{-1}(x)$ has cardinality 2 for all $x \in X$)?

(b) Is there a covering space $p : E \to X$ with $E$ path connected and locally path connected, such that $p$ has 3 sheets?

**Solution.** (a) If we had such a covering space then $H = p_{*}(\pi_1(E, e_0))$ would be a subgroup of $\mathbb{Z}/3\mathbb{Z}$ of index 2, which cannot exist since 2 doesn’t divide 3.

(b) We can take $E$ the universal covering of $X$. To see that this exists, since $X$ is clearly path connected, it’s enough to show that $X$ is locally simply connected, i.e. every point $x \in X$ has a simply connected open neighborhood. This is obvious for all $x$ in the “interior”, while when $x$ is on the boundary, we can construct an open neighborhood of $x$ by taking the 3 rotated copies of $x$ and taking 3 neighborhoods of these copies going into the interior, which after identification become an open neighborhood of $x$ which can be deformation retracted onto a point.