1) Prove or disprove the following “Borsuk-Ulam” theorem for $T^2 = S^1 \times S^1$: given any continuous map $f : T^2 \rightarrow \mathbb{R}^2$, there is a point $(x_1, x_2) \in T^2$ such that $f(x_1, x_2) = f(-x_1, -x_2)$.

**Solution.** This is not true: just take $f(x_1, x_2) = x_1 \in S^1 \subset \mathbb{R}^2$.

2) Let $b_0 = (1, 0) \in S^1 \subset \mathbb{R}^2$. Using the isomorphism $\phi : \pi_1(S^1, b_0) \rightarrow \mathbb{Z}$, that we gave in class, show that every group homomorphism $F : \mathbb{Z} \rightarrow \mathbb{Z}$ can be written as

$$F = \phi \circ f_*) \circ \phi^{-1},$$

for some continuous map $f : (S^1, b_0) \rightarrow (S^1, b_0)$.

**Solution.** $F$ is determined by its value on a generator, so if we denote by $n := F(1) \in \mathbb{Z}$, then we have $F(m) = mn$ for all $m \in \mathbb{Z}$. Given any $m \in \mathbb{Z}$, using complex numbers let $f_m(z) = z^m$, or in real coordinates

$$f_m((\cos(2\pi t), \sin(2\pi t)) = ((\cos(2\pi mt), \sin(2\pi mt)),$$

so that $f_m : (S^1, b_0) \rightarrow (S^1, b_0)$ and $\phi([f_m]) = m$ (as we saw in class). Then we have

$$(f_*) \circ \phi^{-1}(m) = f_*(\phi(m)) = [f_\circ f_m],$$

while $f_\circ f_m(z) = (z^m)^n = z^{mn} = f_{mn}(z)$, and so

$$(f_*) \circ \phi^{-1}(m) = [f_{mn}] = \phi^{-1}(mn),$$

and

$$F(m) = mn = (\phi \circ f_*) \circ \phi^{-1}(m)$$

for all $m \in \mathbb{Z}$ as required.

3) Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a continuous map, and let $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ be the induced homomorphism.

(a) Find an example where $f$ is surjective but not injective, and $f_*$ is injective but not surjective.

(b) Find an example where $f$ is injective but not surjective, and $f_*$ is surjective but not injective.

**Solution.** (a) Take $X = Y = S^1$ and $f(z) = z^2$. This map wraps twice around the circle, counterclockwise, and it is surjective but not injective. After the usual identification $\pi_1(S^1, b_0) = \mathbb{Z}$, we have $f_*(m) = 2m$, which is injective but not surjective.

(b) Take $X = S^1, Y = B^2$, and $f : X \rightarrow Y$ the usual inclusion of the circle as the boundary of the disc. Then $f$ is injective but not surjective, while $f_*$ is the trivial homomorphism from $\mathbb{Z}$ to $\{e\}$, which is surjective but not injective.

4) Let $X$ be the subspace of $\mathbb{R}^2$ given by the union of the two closed unit disc with centers $(1, 0)$ and $(-1, 0)$ (these two discs touch exactly at the origin), and $A \subset X$ be the “figure eight” subspace of $X$ given by the union of the two unit circles with the same centers (these two circles also touch exactly at the origin).
(a) Show that $X$ is simply connected.

(b) Show that $A$ retracts onto $S^1$ (the unit circle with center $(1,0)$, say)

(c) Show that $\pi_1(A,0)$ is nontrivial, where $0$ is the origin.

(d) Is there a retraction $r : X \to A$?

Solution. (a) $X$ is star-shaped around the origin.
(b) The retraction map $\hat{r} : A \to S^1$ is just given by the identity on the right circle, and the constant map on the left circle (which maps it all to the origin).
(c) We have that $\hat{r}_* : \pi_1(A,0) \to \pi_1(S^1) \cong \mathbb{Z}$ is surjective.
(d) We would have $r_* : \pi_1(X,0) \to \pi_1(A,0)$ a surjective homomorphism from the trivial group to a nontrivial one.