Solve all the problems below. Each problem is worth 10 points.

1) Let $B^2 = \{ z \in \mathbb{R}^2 \mid |z| \leq 1 \}$ and let $n \in \mathbb{N}_{>0}$. Let $X = B/ \sim$ where every $z \in \partial B^2$ satisfies $z \sim e^{2\pi i/n} z$ (i.e. the point obtained by rotating $z$ by the angle $2\pi/n$ counterclockwise). Using Seifert-van Kampen, compute $\pi_1(X, b_0)$, where $b_0$ is any point of $X$.

**Solution.** Let $U$ be the open set in $X$ which is the image of $\{ |z| < \frac{3}{4} \}$ and $V$ the open set which is the image of $\{ \frac{1}{2} < |z| \leq 1 \}$. Then $X = U \cup V$, $U$ and $V$ are path connected with $U$ contractible, $V$ deformation equivalent to $\{ |z| = 1 \}/ \sim$, which is homeomorphic to $S^1$, and $U \cap V$ is the image of $\{ \frac{1}{2} < |z| < \frac{3}{4} \}$ hence is path connected and deformation equivalent to $S^1$. Let $\gamma(t) = \frac{2\pi it}{3}, t \in I$, a loop in $U \cap V$ based at $b_0$ (given by $z = \frac{3}{4}$). Then $[\gamma]$ generates $\pi_1(V, 0)$ and $\pi_1(U \cap V, z_0)$ and if $i : \pi_1(U \cap V, z_0) \to \pi_1(V, z_0), j : \pi_1(U \cap V, z_0) \to \pi_1(U, z_0)$ are the homomorphism induced by the inclusions then we have that $j$ is the trivial homomorphism while $i([\gamma])$ is the homotopy class of the path $\gamma$ in $V$, which is the class of the image of the loop in $\partial B^2$ which circles once counterclockwise, and this becomes the class of $n[\gamma]$ in $\pi_1(V, z_0)$. In other words, we have $i([\gamma])j([\gamma])^{-1} = n[\gamma]$. By Seifert-van Kampen we have

$$\pi_1(X, b_0) \cong (\pi_1(U, b_0) \star \pi_1(V, b_0))/N,$$

where $N$ is the normal subgroup generated by $n[\gamma]$, and so

$$\pi_1(X, b_0) \cong \mathbb{Z}/n\mathbb{Z},$$

with the usual understanding that $0\mathbb{Z} = \{0\}$.

2) Let $X \subset \mathbb{R}^2$ be the space given by $X = \bigcup_{n=1}^\infty C_n$, where $C_n$ is the circle in $\mathbb{R}^2$ with radius $n$ and center $(n, 0)$. Compute $\pi_1(X, b_0)$, where $b_0$ is the origin.

**Solution.** Let $U_n$ be the annulus around $C_n$ in $\mathbb{R}^2$, of thickness $\frac{1}{2}$, i.e. $U_n = \{ x \in \mathbb{R}^2 \mid n - \frac{1}{2} < |x - c_n| < n + \frac{1}{2} \}$, where $c_n = (n, 0)$. Then $U_n$ deformation retracts onto $C_n$, so $\pi_1(U_n, b_0) \cong \mathbb{Z}$, and the intersection of any two or three distinct $U_n$'s is a contractible open neighborhood of $b_0$. By Seifert-van Kampen

$$\pi_1(X, b_0) \cong \bigstar_{n \geq 1} \mathbb{Z}.$$

3) Prove that $S^1$ is a retract of $S^1 \vee S^1$ but not a deformation retract.

**Solution.** The retraction map is just given by identity on the left hand circle, and shrinks all the right hand circle to the basepoint.

If $S^1$ was a deformation retract of $S^1 \vee S^1$, these spaces would have the same fundamental group. But we know that the first space has fundamental group $\mathbb{Z}$ (which is abelian) while the second one $\mathbb{Z} \star \mathbb{Z}$ (which is not abelian).

4) Let $X$ be the topological space obtained from the (filled) triangle by identifying all of its three sides as shown:

![Diagram](image-url)
Calculate its fundamental group.

**Solution.** Let $U$ be the open set in $X$ which is the image of almost all the interior of the triangle, and $V$ the open set which is the image of a small neighborhood of the boundary triangle, so that $X = U \cup V$, $U$ and $V$ are path connected, $U$ is contractible, $V$ is deformation equivalent to the image of the boundary triangle, which is homeomorphic to $S^1$, and $U \cap V$ is path connected and deformation equivalent to $S^1$. Let $\gamma$ be a loop in $U \cap V$ which is homotopic to the loop which goes along the boundary triangle once clockwise. Then $[\gamma]$ generates $\pi_1(V, z_0)$ and $\pi_1(U \cap V, z_0)$ and if $i : \pi_1(U \cap V, z_0) \to \pi_1(V, z_0)$, $j : \pi_1(U \cap V, z_0) \to \pi_1(U, z_0)$ are the homomorphism induced by the inclusions then we have that $j$ is the trivial homomorphism while $i([\gamma])$ is the homotopy class of the path $\gamma$ in $V$, which is the class of the image of the loop which goes along the boundary triangle once clockwise, and this becomes the class of $[\gamma] * [\gamma] * [\overline{\gamma}]$ in $\pi_1(V, z_0)$. In other words, we have $i([\gamma])j([\gamma])^{-1} = [\gamma]$. By Seifert-van Kampen we have

$$\pi_1(X, b_0) \cong (\pi_1(U, b_0) \ast \pi_1(V, b_0))/N,$$

where $N$ is the normal subgroup generated by $[\gamma]$, and so

$$\pi_1(X, b_0) \cong \mathbb{Z}/\mathbb{Z} = \{e\},$$

and so $X$ is simply connected.