Solve all the problems below. Each problem is worth 10 points.

1) Let $f : B^2 \to \mathbb{C}$ be a continuous map such that $f(x) \neq 0$ for all $x \in S^1 \subset B^2$. Define a continuous map $g : (S^1, 1) \to (S^1, 1)$ (where we are viewing $S^1 \subset \mathbb{C}$, so 1 is the same as the point $(1, 0) \in S^1 \subset \mathbb{R}^2$), by

$$g(x) = \frac{f(x)}{f(1)}.$$

(a) Show that if $[g] \neq 0$ in $\pi_1(S^1, 1)$ then $f$ must have a zero in $B^2$.

(b) Find an example where $[g] = 0$ in $\pi_1(S^1, 1)$ and $f$ has a zero in $B^2$.

(c) Find an example where $[g] = 0$ in $\pi_1(S^1, 1)$ and $f$ does not have a zero in $B^2$.

**Solution.** (a) If $f$ has no zeros then we can define a continuous map $G : S^1 \times I \to S^1$ by

$$G(x, t) = \frac{f(tx)}{f(t)}.$$

which satisfies $G(x, 0) = 1, G(x, 1) = g(x), G(1, t) = 1$, and so is a path homotopy between $g$ and the constant loop.

(b) $f(z) = |z|$.

(c) $f(z) = 1$.

2) In the same setting as problem 1, suppose $[g] \neq 0$ in $\pi_1(S^1, 1)$ (so that $f$ must have a zero in $B^2$ by problem 1(a)). Given $h : B^2 \to \mathbb{C}$ a continuous map such that

$$|h(x)| < |f(x)|,$$

for all $x \in S^1$, then $f + h$ must have a zero in $B^2$.

**Solution.** For any $x \in S^1$ we have

$$|f(x) + h(x)| \geq |f(x)| - |h(x)| > 0,$$

so $f + h$ does not vanish on $S^1$. Next, we show that $k : S^1 \to \mathbb{C}$ given by

$$k(x) = \frac{f(x) + h(x)}{|f(1) + h(1)|},$$

satisfies $[k] \neq 0$ in $\pi_1(S^1, 1)$. Indeed, we show that $[k] = [g]$ via the path homotopy $G : S^1 \times I \to S^1$,

$$G(x, t) = \frac{f(x) + th(x)}{|f(1) + th(1)|},$$

which makes sense because

$$|f(x) + th(x)| \geq |f(x)| - t|h(x)| \geq |f(x)| - |h(x)| > 0,$$
for all $x \in S^1$. Thus $f + h$ has a zero in $B^2$ by problem 1(a).

3) Is there a retraction $r : X \to A$ where $X = S^1 \times B^2$ and $A = S^1 \times S^1 \subset X$ where $S^1 \subset B^2$ is the usual inclusion of the circle as the boundary of the disc?

**Solution.** Let $b_0$ be a point in $A$. We have $\pi_1(X, b_0) \cong \pi_1(S^1) \times \pi_1(B^2) \cong \mathbb{Z}$ (where we pick any basepoints on $S^1$ and $B^2$), while $\pi_1(A, b_0) \cong \mathbb{Z}^2$. If we had a retraction $r : X \to A$ then $r_* : \pi_1(X, b_0) \to \pi_1(A, b_0)$ would be surjective, and so we would obtain a surjective group homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}^2$. But if we set $\phi(1) = (m_1, m_2)$ then $\phi(n) = (nm_1, nm_2)$ and this cannot be surjective because it misses all points $(a, b)$ in $\mathbb{Z}^2$ with $a$ and $b$ coprime.

4) Is there a retraction $r : X \to A$ where $X = S^1 \times B^2$ and $A \subset X$ is the subspace pictured below, which is homeomorphic to a circle $S^1$?

![Diagram](image.png)

**Solution.** Dropping basepoints from the notation, we have that $\pi_1(X) \cong \pi_1(S^1) \cong \mathbb{Z}$, and $\pi_1(A) \cong \pi_1(S^1) \cong \mathbb{Z}$, generated by $[f]$ where $f$ is the loop that traces out $A$. If $j : A \to X$ is the inclusion then $j_*([f]) = [j \circ f]$ and we claim that the loop $j \circ f$ is homotopic to zero in $X$: indeed if $\pi : X \to S^1$ is the projection to the second factor, then $\pi_* : \pi_1(X) \to \pi_1(S^1)$ is an isomorphism (since $\pi_1(B^2) = 0$) so it is enough to check that $\pi_*([j \circ f]) = 0$ in $\pi_1(S^1)$, and this equals $[\pi \circ j \circ f]$; from the picture, $\pi \circ j \circ f$ a loop in $S^1$ which winds once clockwise and then once counterclockwise, and so it is homotopic to zero as we know. Therefore $j_*$ is the trivial homomorphism which is absurd since $r : X \to A$ is a retraction.