Lecture 1

Complex $C^\cdot$ of vector spaces is <u>perfect</u> if $\dim H^\cdot(C^\cdot) < \infty$

To a perfect $C^\cdot$:

$$\chi(C^\cdot) = \sum_{k\in\mathbb{Z}} (-1)^k \dim H^k(C^\cdot) \in \mathbb{Z}$$

$$\det(C^\cdot) = \bigotimes_{k\in\mathbb{Z}} \left(\Lambda^{top} H^k(C^\cdot)\right)^{\otimes(-1)^k}$$

(the determinant line)

Examples

<u>Df</u>: a complex of A-modules is perfect if it is quasi-isom to a finite complex of fin. gen. proj. mods

a complex of sheaves of mods / sheaf of algs $\mathcal{A}$ <u>perfect</u> if <u>locally</u> perfect.

Ex. 1 A constructible sheaf $\mathcal{F}$ on a (compact) manifold $\rightsquigarrow C^\bullet(\mathcal{F}) = \mathbb{R}\Gamma(\mathcal{F})$
$\rightsquigarrow H^\bullet(X, \mathcal{F})$

Ex. 2 A (coherent or perfect) sheaf of $\mathcal{D}_X$-modules $\mathcal{M}$ on a complex (compact) manifold.
$C^\bullet = \mathbb{R}\Gamma(\mathcal{DR}^\bullet(\mathcal{M}))$, the De Rham complex of $\mathcal{M}$.

Ex. 3 A pair $(\mathcal{M}, \mathcal{F})$: $\mathcal{M}$ a $\mathcal{D}_X$-module with a good filtration F_i as in Ex. 2
$\mathcal{F}$ a ($\mathbb{R}$-)constructible sheaf as in Ex. 1,
X may not be compact. Condition:

$$SS(\mathcal{M}) \cap SS(\mathcal{F}) \subset T^*_X X, \text{ compact.}$$

Schapira - Schneiders: $\boxed{\mathbb{R}\Gamma(\mathcal{DR}^\bullet(\mathcal{M}) \otimes_{\mathbb{C}} \mathcal{F}) \text{ perf.}}$

Partial cases:

a) $\mathcal{M} = \mathcal{O}_X$; $\mathcal{F}$ a sheaf ; get $R\Gamma(X, \mathcal{F})$

b) $\mathcal{F} = \mathbb{C}_X$; $\mathcal{M}$ $\mathcal{D}_X$-mod; get $R\Gamma(X, DR^{\cdot}(\mathcal{M}$

c) $X_{\mathbb{R}}$ compact real analytic manifold ; $X = X_{\mathbb{C}}$ - complexification of $X_{\mathbb{R}}$;

$\mathcal{E} \xrightarrow{D} \mathcal{F}$ - real analytic diff. op., elliptic, on $X_{\mathbb{R}}$

~~$\mathcal{M} = \mathcal{E}_{\mathbb{C}} \xrightarrow{D_{\mathbb{C}}} \mathcal{F}_{\mathbb{C}}$, the complex~~

$$\mathcal{M} = Dif(\mathcal{O}_{X_{\mathbb{C}}}, \mathcal{E}_{\mathbb{C}}) \xrightarrow{\cdot D_{\mathbb{C}}} Dif(\mathcal{O}_{X_{\mathbb{C}}}, \mathcal{F}_{\mathbb{C}})$$

$$\mathcal{F} = \mathbb{C}_{X_{\mathbb{R}}}$$

$$\boxed{DR^{\cdot}(\mathcal{M}) \otimes_{\mathbb{C}} \mathcal{F} = (\mathcal{E} \xrightarrow{D} \mathcal{F})}$$

d) $\mathcal{E}$ - holom ~~vector~~ bundle on X;

$$\mathcal{M} = \mathcal{D}if(\mathcal{O}_X, \mathcal{E}); \quad DR(\mathcal{M}) = \mathcal{E};$$

$$\mathcal{F} = \mathbb{C}_X; \quad DR(\mathcal{M}^\cdot) \otimes_{\mathbb{C}} \mathcal{F} = R\Gamma(X, \mathcal{E})$$

$\equiv$

Microlocal formulas for $\chi(C^\cdot)$ from examples 1-3:

$$\chi(C^\cdot) = \int_{T^*X} \left(\begin{array}{l} \text{some char. class} \\ \text{w/ compact support} \end{array} \right)$$

$\Downarrow$

RR for $\chi(\mathcal{E})$;

Atiyah-Singer for index (D);

Brylinski-Kashiwara-Dubson formulas for $\chi(X, \mathcal{F})$;

$\chi(X, DR^\cdot(\mathcal{M}))$;

...

For examples 1-3:

$$\det(C^\bullet) = ?$$

What does it mean to compute a line?

(all lines are isomorphic...)

a)
Data $D \rightsquigarrow$ line $\mathcal{L}_D$

Data $D, E \rightsquigarrow$ isom $g_{DE} : \mathcal{L}_D \xrightarrow{\sim} \mathcal{L}_E$

$$g_{DE}\, g_{EF} = g_{EF}$$

($\approx$ a line bundle on the set of data)

b) Sometimes complexes come in families; $S \ni s \rightsquigarrow$ complex $C_s^\bullet$

$s \mapsto \det(C_s^\bullet)$ - a line bundle on S.

Ex: $X \xrightarrow{f} S$ $\quad s \mapsto R\Gamma(X_s, \mathcal{E}_s) \to \det R\Gamma(X_s, \mathcal{E}_s)$

($\mathcal{E}$ - holom bundle on X)

Can be interpreted as $c_1(f_* \mathcal{E})$ and computed by Riemann-Roch.

[Does D. Patel's computation of $R\Gamma(DR^\bullet(\mathcal{M}))$ recover this?..

c) Computation of the line $\det(R\Gamma(X, DR(\mathcal{M}))$ does actually lead to facts about numbers.

$X\,;\ \mathcal{M}\in\mathcal{D}_X\text{-mod} \rightsquigarrow \mathcal{F} := DR(\mathcal{M})\underset{\mathbb{Q}}{\otimes}\mathbb{C}$

Say, X and $\mathcal{M}$ algebraic, defined / $\mathbb{Q}$

a complex of sheaves

$\det \mathbb{R}\Gamma(DR^\bullet(\mathcal{M}))$

a line over $\mathbb{Q}$

Sometimes: $\mathcal{F}$ carries its own $\mathbb{Q}$ structure

Example: $\mathcal{F}$ happens to be constant ($\mathcal{M}=\mathcal{O}_X$); or a local system with rational monodromy, or...

The period map:

$$R\Gamma(\mathcal{DR}^{\cdot}(\mathcal{M})) \xrightarrow{\sim} R\Gamma(\mathcal{F}_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$$

Its determinant:

$$\overset{\mathrm{top}}{\Lambda} R\Gamma(\mathcal{DR}^{\cdot}(\mathcal{M})) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \Lambda^{\mathrm{top}} R\Gamma(\mathcal{F}_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$$

a well-defined element of $\mathbb{C}^{\times}/\mathbb{Q}^{\times}$

(indep. of a choice of a $\mathbb{Q}$-basis).

In terms of what data do we compute the line $\mathcal{L}$?

$\chi(C^\cdot)$, $C^\cdot = R\Gamma(X, \mathcal{F})$ or $R\Gamma(X, DR^\cdot(\mathcal{M}$

in terms of a cohomology class

$$c \in H^\cdot_Z(T^*X)$$

Microlocal Euler class of Kashiwara-Schapira

$\det(C^\cdot)$ as well (but the cohomology is different). We pull c back on X by a one-form ν:

$$\nu : X \underset{\longrightarrow}{\overset{\pi}{\curvearrowleft}} T^*X \qquad \pi\nu = id_X$$

$$\nu^*(c) \in H^*_{\nu^{-1}Z}(X)$$

If $\#\{x \mid \nu(x) \in Z\}$ is finite, then

$$\boxed{\det(C^\cdot) = \mathcal{L} = \bigotimes_{\nu(x)\in Z} \mathcal{L}_\nu}$$

$$\begin{tikzcd}
\det R\Gamma(X, DR^{\bullet}(\mathcal{M})) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r, "\simeq", "\det(\text{period map})"'] \arrow[d] & \det R\Gamma(X, \mathcal{F}_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[d] \\
\bigotimes_{x:\, v(x) \in \mathbb{Z}} \mathcal{L}_v \arrow[r, "\sim"] & \bigotimes_{x \in v(x) \in \mathbb{Z}} \mathcal{L}'_v
\end{tikzcd}$$

Does the bottom horizontal map split into a product?

Structure of a formula:

$$\boxed{\det\left(\int_{\gamma_i} \omega_j\right) = \prod_{x:\, v(x) \in \mathbb{Z}} d_v}$$

Subject of S. Beilinson's lectures

<u>Example:</u>

$$\int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

$(x = 0, 1, \infty;\ X = \mathbb{P}^1)$

Regularized determinants

$$D : H_+ \to H_- \qquad \text{Fredholm}$$

Intuitively:

$$\Lambda^{max} D \in \Lambda^{max}\left(H_+/_{\ker D}\right)^* \otimes \Lambda^{max}(\operatorname{im} D)$$

$$\cong$$

$$\left(\Lambda^{max} H_+\right)^* \otimes \Lambda^{max} H_- \otimes \left(\Lambda^{max} \ker D \Big/ \left(\Lambda^{max} \operatorname{coker} D\right)^*\right)$$

a family $D_s : H_+ \to H_-$ $\qquad \Big\| \qquad$ $s \in S$ $\qquad \operatorname{Det}(D_s)$

$$s \mapsto \Lambda^{max} \ker D_s \Big/ \left(\Lambda^{max} \operatorname{coker} D_s\right)^*$$

can be made a line bundle on S

Topological obstruction: this line bundle can be nontrivial.

If it trivializes:

$$\operatorname{Det}(D_s) \xrightarrow{\simeq} \mathbb{C}$$

regularized determinant:

An element of $\Lambda^{max} \ker / (\Lambda^{max} \operatorname{coker})^*$

becomes now a <u>number</u>.

<u>Example</u> (Quillen):

$$\det_\zeta(DD^*) = e^{-\zeta'_{DD^*}(0)}$$

where $\zeta_{DD^*}(s) = \sum_{\substack{\lambda \neq 0 \\ \lambda \in \operatorname{Spec}(DD^*)}} \lambda^{-s}$

Its natural interpretation:
it is a <u>metric</u> on the line bundle
$\mathcal{L} = (\Lambda \ker)^* \otimes \Lambda \operatorname{coker}$, i.e. an element of
$\mathcal{L}^* \otimes \bar{\mathcal{L}}^*$. When the line bundle trivializes:

$$D_{reg}(DD^*) = (\text{factor}) \cdot \det_\zeta(DD^*)$$

$\equiv$

To what extent are there analogies
between regularized determinants and
determinants of period matrices? Are
there product formulas?

Lecture 2

DG category: $i, j, \ldots$ - <u>objects</u>;

$C^\bullet(i,j)$ - a complex; $C^\bullet(i,j) \otimes C^\bullet(j,k) \to$

$\to C^\bullet(i,k)$

morphisms of complexes;

associative, with units.

Intuition: DG category with one object - DGA;

an algebra = an algebra of <u>functions</u>;

a DG category = category of complexes of "vector bundles" on some (NC) space.

Invariants of a DG category:

DG cat. $A \rightsquigarrow$ <u>complex</u> $C^\bullet(A)$

or, more generally, a <u>spectrum</u>

$\downarrow$

cohomology groups of $C^\bullet(A)$

(or, more generally, π_i of a spectrum)

A complex: $C^\bullet$; $C^\bullet \xrightarrow{d} C^{\bullet+1}$; $d^2=0$

A spectrum $\mathbb{P}$: top. spaces (pointed) P_n

$$S^1 \wedge P_n \to P_{n+1}$$

Homotopy groups of $\mathbb{P}$:

$$\pi_i(\mathbb{P}) = \varinjlim_{n\to\infty} \pi_{i+n}(P_n)$$

A complex $C^\bullet$ gives rise to a spectrum $|C^\bullet|$:

$$\mathbb{P}_n = |C^\bullet[n]|$$

(for any complex $C^\bullet$, $|C^\bullet|$ is the Dold-Kan realization:

$$n \longmapsto \operatorname{Hom}_{\text{Complexes}}(C_{-\bullet}(\Delta^n), C^\bullet)$$

$$\pi_i(|C^\bullet|) = H^{-i}(C^\bullet)$$

DG category $\mathcal{A}$

$\downarrow$

<u>Complexes</u> $C_\bullet(\mathcal{A})$; $CC_\bullet^-(\mathcal{A})$; $CC_\bullet^{per}(\mathcal{A})$

Hochschild ; negative cyclic ; periodic cyclic

<u>Spectra</u> $K(\mathcal{A})$

Also, for a Banach or Fréchet algebra,

$K^{top}(A)$.

Definitions and properties - later.

<u>Properties</u>:

Object i of $\mathcal{A}$ $\rightsquigarrow$ point $[i]$ of $K(\mathcal{A})$ (<u>homotopy</u> point)

Chern character: $K(\mathcal{A}) \xrightarrow{ch} |CC_\bullet^-(\mathcal{A})|$

A DG functor $\mathcal{A} \to \mathcal{B}$ induces morphisms of complexes/spectra $C_\bullet(\mathcal{A}) \to C_\bullet(\mathcal{B})$, etc.

A weak equivalence induces a weak equivalence.

(A weak equivalence of DG categories: $\mathcal{A} \xrightarrow{F} \mathcal{B}$ DG functor; $\mathcal{A}^\bullet(i,j) \to \mathcal{B}^\bullet(Fi, Fj)$ quasi-isomorphisms; F induces equivalence of <u>homotopy categories</u> $\mathrm{Ho}(\mathcal{A}^\bullet) = H^0(\mathcal{A}^\bullet(-,-))$)

A weak equivalence of complexes/spectra induces an isomorphism of H^i / π_i.

In particular: an <u>A_∞ functor</u>

$$\begin{tikzcd} & \mathcal{C}^\bullet \arrow[dl, "\text{quis}"'] \arrow[dr] & \\ \mathcal{A}^\bullet & & \mathcal{B}^\bullet \end{tikzcd}$$

induces morphisms $K_i(\mathcal{A}^\bullet) \to K_i(\mathcal{B}^\bullet)$, etc.

$\mathrm{Perf}(k)$ = DG category of perfect complexes of vector spaces / k

$$K(\mathrm{Perf}(k)) \longrightarrow \mathrm{Nerve}(\mathrm{Pic}(k))$$

$\mathrm{Pic}(k)$ = groupoid : objects = lines/ k
morphisms = isomorphisms

$$HC_0^-(\mathrm{Perf}(k)) \to k$$

$$\mathrm{ch}(C^\bullet) \longmapsto \chi(H^\bullet(C^\bullet))$$

More precisely: an A_∞ functor

$$\cancel{HC_0^-}\,\mathrm{Perf}(k) \to s\,\mathrm{Perf}(k)$$

Strictly perfect complex is a finite complex of fin. dim. v. spaces.

Microlocal approach to $\mathbb{R}\Gamma(\mathcal{DR}^\bullet(\mathcal{M}))$

All our $\mathcal{D}_X$-modules are <u>perfect</u>.

$F\mathcal{D}_X$-mod = DG category of (perfect) complexes of $\mathcal{D}_X$-modules w/ good filtr.

$$\mathcal{M} \longmapsto \mathrm{Rees}(\mathcal{M}) = \bigoplus_{n \geq 0} \hbar^n \cdot F_n\mathcal{M}$$

$$ev_{\hbar=1} : \mathrm{Rees}(\mathcal{D}_X) \to \mathrm{Rees}(\mathcal{D}_X)/(\hbar - 1) = \mathcal{D}_X$$

$$\mathrm{Rees}(\mathcal{M}) \longrightarrow \mathcal{M}$$

polyn. in

$$ev_{\hbar=0} : \mathrm{Rees}(\mathcal{D}_X) \to \mathrm{Rees}(\mathcal{D}_X)/(\hbar) = \mathcal{O}_{T^*X}$$

$$\mathrm{Rees}(\mathcal{M}) \longmapsto \mathrm{gr}\,\mathcal{M}$$

Microlocalization:

$$F\mathcal{D}_X\text{-mod} \longrightarrow \pi^{-1}\mathrm{Rees}\,\mathcal{D}_X\text{-mod} \xrightarrow[ev_{\hbar=0}]{} \mathcal{O}^{poly}_{T^*X}\text{-mod}$$

$\boxed{\text{(principal symbol)}}$

D. Patel's 'Brylinski-Dubson-Kashiwara formula'

$$
\begin{tikzcd}
\mathcal{D}_X\text{-module } \mathcal{M}, F_\bullet \arrow[r, squiggly] \arrow[d, squiggly] & \mathcal{O}^{poly}_{T^*X}\text{-module } \sigma(\mathcal{M}) \arrow[d, squiggly] \\
\begin{array}{c}\text{Complex of sheaves}/\mathbb{C}_X \\ \mathcal{DR}^\bullet(\mathcal{M})\end{array} \arrow[d, squiggly] & \begin{array}{c}\text{Complex of sheaves}/\pi^{-1}\mathcal{G} \\ \pi^{-1}\omega_X \overset{L}{\otimes}_{\mathcal{O}^{poly}_{T^*X}} \sigma(\mathcal{M})\end{array} \arrow[d, squiggly] \\
\begin{array}{c}\text{Perfect complex of} \\ \text{vector spaces}/\mathbb{C} \\ R\Gamma(X, \mathcal{DR}^\bullet(\mathcal{M}))\end{array} \arrow[d, squiggly] & \begin{array}{c}C^\bullet(\mathcal{M}) := \nu^{-1}\left(\pi^{-1}(\omega_X) \overset{L}{\otimes}_{\mathcal{O}^{poly}_{T^*X}} \sigma(\mathcal{M})\right) \\ \text{Complex of sheaves} \\ \text{on } X \text{ supported on } \underline{\nu^{-1}Z}\end{array} \arrow[d, squiggly] \\
\boxed{\underline{\text{Line}} \quad \det R\Gamma(X, \mathcal{DR}^\bullet(\mathcal{M})) \xrightarrow[\sim]{\text{Thm}} \bigotimes_{x \in \nu^{-1}Z} \det C^\bullet(\mathcal{M})_x}
\end{tikzcd}
$$

Here: $SS(\mathcal{M}) \subset Z$ $\qquad \nu : X \rightleftarrows T^*X$, $\pi \circ \nu = id_X$

$\nu^{-1}Z$ <u>finite</u> $\qquad$ i.e. $\nu \in \Omega^1(X)$

D. Patel's Brylinski-Dubson-Kashiwara formula: proof (idea)

$$
\begin{tikzcd}
K(F\mathcal{D}_X\text{-mod}) \arrow[r] \arrow[d, "\mathcal{DR}^{\cdot} = \omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} -"] & K(\pi^{-1}\mathrm{Rees}\,\mathcal{D}_X\text{-mod}) \arrow[r, "ev_{\hbar=0}"] \arrow[d, "\pi^{-1}\mathrm{Rees}(\omega_X) \overset{L}{\underset{\pi^{-1}\mathrm{Rees}\,\mathcal{D}_X}{\otimes}} -"] & K(\mathcal{O}^{poly}_{T^*X}\text{-mod}) \arrow[d, "\pi^{-1}\omega_X \overset{L}{\underset{\mathcal{O}_{T^*X}}{\otimes}} -"] \\
K(\mathbb{C}_X\text{-mod}) \arrow[d, equal] & K(\pi^{-1}\mathbb{C}_X[\hbar]\text{-mod}) \arrow[l] \arrow[r, "ev_{\hbar=0}"] \arrow[d, "\nu^{-1}"] & K(\pi^{-1}\mathbb{C}_X\text{-mod}) \arrow[d, "\nu^{-1}"] \\
K(\mathbb{C}_X\text{-mod}) \arrow[dr, "R\Gamma"] & K(\mathbb{C}_X[\hbar]\text{-mod}) \arrow[l, shift right, "ev_{\hbar=0}"'] \arrow[l, shift left, "ev_{\hbar=1}"] \arrow[r, "ev_{\hbar=0}"] & K(\mathbb{C}_X\text{-mod}) \arrow[dl, "R\Gamma"'] \\
 & \mathrm{Perf}(\mathbb{C}) &
\end{tikzcd}
$$

Used rigidity:

$\boxed{ev_{\hbar=0} = ev_{\hbar=1}}$ for K, CC^{per}.

Therefore

$$\det R\Gamma(X, \mathcal{DR}^{\cdot}(\mathcal{M})) \simeq \det R\Gamma\Big(X, \nu^{-1}\big(\pi^{-1}\omega_X \overset{\mathbb{L}}{\underset{\mathcal{O}_{T^*X}}{\otimes}} \sigma(\mathcal{M})\big)\Big)$$

Recall:

$$
\begin{tikzcd}
T^*X \arrow[d, "\pi"] \\
X \arrow[u, bend left, "\sim"]
\end{tikzcd}
$$

($\det R\Gamma(\mathcal{DR}^{\cdot}(\mathcal{M}))$ in terms of the principal symbol of $\mathcal{M}$)

Refine this:

$Z \subset T^*X$; $F\mathcal{D}_X\text{-mod}_Z$ -

subcategory $\{ \mathcal{M} \mid SS(\mathcal{M}) \subset Z \}$.

$$\begin{array}{ccc}
K(F\mathcal{D}_X\text{-mod}_Z) & \longrightarrow & K(\pi^{-1}\mathrm{Rees}\,\mathcal{D}_X\text{-mod}_{\mathrm{supp}\subseteq Z}) \\
 & & \downarrow {\scriptstyle ev_{\hbar=0}} \\
 & & K(\mathcal{O}^{\mathrm{poly}}_{T^*X}\text{-mod}_{\mathrm{supp}\subseteq Z}) \\
 & & \downarrow {\scriptstyle \omega_X \overset{L}{\underset{\mathcal{O}_{T^*X}}{\otimes}} -} \\
 & & K(\pi^{-1}\mathbb{C}_X\text{-mod}_{\mathrm{supp}\subseteq Z}) \\
 & & \downarrow {\scriptstyle v^{-1}} \\
 & & K(\mathbb{C}_X\text{-mod}_{\mathrm{supp}\subseteq v^{-1}(Z)}) \\
 & & \parallel \\
\prod_{x:\, v(x)\in Z} K(\mathbb{C}) & = & K_Z(\mathbb{C}_X\text{-mod})
\end{array}$$

$$\det(R\Gamma(X, DR^{\cdot}(\mathcal{M}))) \simeq \bigotimes_{x:\, v(x)\in Z} \mathcal{L}_x$$

All this, without any change, can be done for $CC^{per}_{\cdot}$ instead of K. Get

$$\chi(DR^{\cdot}(\mathcal{M})) = \sum_{x \in v^{-1}(Z)} \chi_x$$

<u>Example:</u>

$$\chi(DR^{\cdot}(\mathcal{O}_X)) = \sum_{x \in Crit(f)} (-1)^{\mu(f)}$$

for $v = df$, a Morse function.

<u>Microlocal approach to $\det(\mathbb{R}\Gamma(X, \mathcal{F}))$</u>

(Beilinson).

<u>More generalities on spectra.</u>

<u>Verdier duality.</u>

Presheaf of complexes: $U \rightsquigarrow C^{\cdot}(U)$

$U \subset V$: $C^{\cdot}(U) \leftarrow C^{\cdot}(V)$ subject to...

A presheaf is a <u>sheaf</u> if ...

Similarly - presheaves of <u>spectra</u>.

<u>Dually</u>: a precosheaf $\mathcal{B}^\bullet : U \mapsto \mathcal{B}^\bullet(U)$;

$U \hookrightarrow V \quad \mathcal{B}^\bullet(U) \to \mathcal{B}^\bullet(V)$; cosheaf - ...

Precosheaves of spectra.

a presheaf of complexes $\mathcal{F}^\bullet(U) \rightsquigarrow$

$$C^\bullet(X, \mathcal{F}^\bullet) = \lim_{\text{cover } \mathcal{U}} C^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$$

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{U_0, \dots, U_p \in \mathcal{U}} \mathcal{F}^\bullet(U_0 \cap \dots \cap U_p)$$

a precosheaf $\mathcal{B}^\bullet \rightsquigarrow C_\bullet(X, \mathcal{B}^\bullet) = \underleftarrow{\lim}\, C_\bullet(\mathcal{U}, \mathcal{B}^\bullet)$

$$C_p(\mathcal{U}, \mathcal{B}^\bullet) = \bigoplus_{U_0, \dots, U_p} \mathcal{B}^\bullet(U_0 \cap \dots \cap U_p)$$

Similar theories for spectra:

spectra $\quad C^\bullet(X, \mathcal{F}) \quad \mathcal{F}$ - presheaf

$\quad C_\bullet(X, \mathcal{B}) \quad \mathcal{B}$ - precosh. of sp.

(taking some liberty w/ notation)

$$C^0(\mathcal{U},\mathcal{F}) \rightrightarrows C^1(\mathcal{U},\mathcal{F}) \rightrightarrows C^2 \dots$$

$$C^p(\mathcal{U},\mathcal{F}) = \prod_{U_0,\dots,U_p} \mathcal{F}(U_0 \cap \dots \cap U_p)$$

cosimplicial object in spectra

$$C^\bullet(\mathcal{U},\mathcal{F}) = \mathrm{Tot}(p \rightsquigarrow C^p) =$$

$$= \mathrm{Hom}_{\Delta}(\Delta^\bullet, C^\bullet(\mathcal{U},\mathcal{F}))$$

a spectrum.

<u>Dually:</u>

$$C_p(\mathcal{U},\mathcal{B}) = \coprod_{U_0,\dots,U_p} \mathcal{B}(U_0 \cap \dots \cap U_p)$$

$$C_\bullet(\mathcal{U},\mathcal{B}) = |p \rightsquigarrow C_p| =$$

$$= \coprod \Delta^n \times C_n(\mathcal{U},\mathcal{B}) / \sim$$

Sheaf $\mathcal{F} \rightsquigarrow$ Verdier dual $\mathbb{D}\mathcal{F}$

$$\mathbb{D}\mathcal{F}(U) = \Gamma_c(U, \mathcal{F})$$

(the correct duality functor is derived to that; i.e., strictly speaking,

$$\mathbb{D}\mathcal{F}(U) = \mathbb{R}\Gamma_c(U, \mathcal{F}) =$$

$$= \Gamma_c(U, \mathcal{J}^\bullet)$$

$\mathcal{J}^\bullet$ – injective resolution of $\mathcal{F}$)

Note:

$$\mathbb{D}\mathcal{F}(U) = \Gamma_c(U, \mathcal{F}) = \varinjlim_{K \Subset U} \ker\Big(\Gamma(U, \mathcal{F}) \to \Gamma(U \setminus K, \mathcal{F})\Big)$$

dually:

$$\mathbb{D}\mathcal{B}(U) = \varprojlim_{K \Subset U} \operatorname{coker}\Big(\Gamma(U, \mathcal{B}) \leftarrow \Gamma(U \setminus K, \mathcal{B})\Big)$$

For $\mathcal{F}(U) = \{(a_x \in A_x) \mid x \in U\}$

$$\mathbb{D}\mathbb{D}\mathcal{F} = \mathcal{F}$$

But any $\mathcal{F}$ has an injective resolution of this form. Therefore

$$\mathbb{D}\mathbb{D} = id$$

Similarly for cosheaves.

<u>Verdier duality for presheaves / precosheaves of spectra: similar.</u>

$$\mathbb{D}\mathcal{F}(U) = R\Gamma_c(U, \mathcal{F}) =$$

$$= \varinjlim_{K \Subset U} \operatorname{fibre}\left(C^\bullet(U, \mathcal{F}) \to C^\bullet(U - K, \mathcal{F})\right)$$

...

Lecture 3

$\mathrm{Sheaves}_Z(X)$ = Constructible sheaves of R-mod on X :
$$SS(\mathcal{F}) \subset Z$$

Beilinson's construction seems to be:

$$\begin{tikzcd}
K(\mathrm{Sheaves}(X)) \arrow[r] & C_\bullet(X, K(R)) \arrow[d] \\
K(\mathrm{Sheaves}(X)_Z) \arrow[u] \arrow[d, dashed, "!"'] & C^\bullet(X, \mathbb{D}K(R)) \arrow[d] \\
C^\bullet_Z(T^*X, \pi^{-1}\mathbb{D}K(R)) \arrow[r] & C^\bullet(T^*X, \pi^{-1}\mathbb{D}K(R)) \\
C^\bullet_{\nu^{-1}Z}(X, \mathbb{D}K(R)) \arrow[u, "\nu^{-1}"'] &
\end{tikzcd}$$

constant cosheaf

$$C^\bullet_{\nu^{-1}Z}(X, \mathbb{D}K(R)) = \prod_{x \in \nu^{-1}(Z)} C^\bullet_{\{x\}}(V, \mathbb{D}K(R)) = \prod_{x \in \nu^{-1}(Z)} K(R)$$

An identical procedure if K is replaced by the Hochschild complex $C_\bullet(\mathbb{C})$, for a sheaf of $\mathbb{C}$-vector spaces, gives

$$\mu eu(\mathcal{F}) \in H^{2n}_{Z}(T^*X, \mathbb{C})$$

(almost for sure equivalent to the microlocal Euler class of Kashiwara-Schapira).

$$\chi(R\Gamma(X,\mathcal{F})) = \int_{T^*X} \mu eu(\mathcal{F}) \cup [T^*_X X]$$

Poincaré dual

(Dubson-Kashiwara formula)

Microlocal approach to det, χ of $R\Gamma(DR^{\cdot}(\mathcal{M}) \underset{\mathbb{C}}{\otimes} \mathcal{F})$ when

$$SS(\mathcal{M}) \cap SS(\mathcal{F}) \subset T^*_X X$$

compact

(Schapira - Schneiders elliptic pairs)

Back to microlocalization.

$$F\mathcal{D}_X\text{-mod} \longrightarrow \pi^{-1}\mathrm{Rees}\,\mathcal{D}_X\text{-mod}$$

$$\mathcal{M} \longmapsto \pi^{-1}\mathrm{Rees}\,\mathcal{M}$$

Deformation quantization of T^*X:

$$\begin{array}{ccc} \pi^{-1}\,\mathrm{Rees}\,\mathcal{D}_X & \hookrightarrow & \mathcal{O}^{\hbar}_{T^*X} \\ \text{locally}\ \wr\wr & & \wr\wr\ \text{locally} \\ \mathcal{O}^{\text{poly in }\xi}_{T^*X}[\hbar] & \hookrightarrow & \mathcal{O}_{T^*X}[[\hbar]] \end{array}$$

(because the local products and gluing isoms ex

Let $\mathcal{A}$ be any sheaf of algebras on a space M.

<u>Descent morphisms:</u>

$$K(\mathrm{Perf}(\mathcal{A}\text{-mod})) \to C^{\bullet}(M, K(s\mathrm{Perf}(\mathcal{A})))$$

presheaf of spectra

$U \mapsto K(s\mathrm{Perf}(\mathcal{A}(U))) = K(\mathcal{A}(U))$

Similar for $C_{\cdot}^{Hoch}$, $CC_{\cdot}^{-}$, ...

Should follow from:

① $\mathrm{Perf}(\mathcal{A}\text{-mod}) \to A_\infty \text{ functors}(\mathrm{Open}(M), s\mathrm{Perf}(\mathcal{A}))$

$\mathrm{Open}(M)$ - category of open subsets of M; morphisms - inclusions.

② $$\mathcal{B} \otimes A_\infty \mathrm{funct}(\mathcal{B}, \mathcal{C}) \xrightarrow{ev} \mathcal{C}$$

DG functor, $\forall$ DG cats $\mathcal{B}, \mathcal{C}$

a realization of this idea for $CC_\bullet^-$, ... : Bressler, Gorokhovsky, Nest, B.T., Chern character of twisted complexes.

- Probably works for K with minor changes
- probably are other, better realizations.

$$K(FD_X\text{-mod}_Z) \to K(\pi^{-1}\text{Rees } D_X\text{-mod}_Z)$$

$$\to K(\mathbb{O}^{\hbar}_{T^*X}\text{-mod}_Z) \to C^\bullet_Z(T^*X, K(\mathbb{O}^{\hbar}_{T^*X}$$

$$=$$

$$\boxed{\begin{array}{l} K(\text{Sheaves}(X)_Z) \longrightarrow C^\bullet_Z(T^*X, \mathbb{D}K(\mathbb{C})) \\ K(FD_X\text{-mod}_{Z'}) \to C^\bullet_{Z'}(T^*X, K(\mathbb{O}^{\hbar}_{T^*X})) \end{array}}$$

similarly for $CC_\bullet^-$, ...

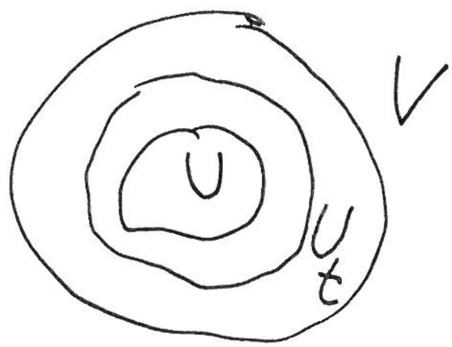
Remark The Beilinson map

$$K(Sh(X)_Z) \to C^{\bullet}_Z(T^*X, \mathbb{D}K(\mathbb{C}))$$

should follow from something like

$$Sh(X)_Z \simeq A_\infty \text{ functors}(\mathrm{Open}(X)_{loc}, \mathrm{Perf}(\mathbb{C}))$$

where $\mathrm{Open}(X)_{loc} = \mathbb{C}$ [cat. of opens in X, localized by $U \hookrightarrow V$:



$$U = \{f_0(x) \geq 0\}$$

$$V = \{f_1(x) > 0\}$$

$$U_t = \{f_t(x) > 0\}$$

$$\nabla f_t \notin Z \quad \text{on } \partial U_t$$

It seems that something similar is mentioned in Nadler-Zaslow.

FACT: After inverting $\hbar$, get quasi-isomorphisms of sheaves:

$$CC_\bullet^-\left(\mathcal{O}^\hbar_{T^*X}[\hbar^{-1}]\right) \xrightarrow{\sim} \mathbb{D}CC_\bullet^-\left(\mathbb{C}_{T^*X}((\hbar))\right)$$

Similarly for $C_\bullet^{Hoch}$, $CC_\bullet^{per}$.

Note that $CC_\bullet^-(k) \simeq k[[u]]$, $|u| = -2$.

Projecting along the ideal (u), get

$$\mu eu^\hbar(\mathcal{M}) \in H^{2n}_Z(T^*X, \mathbb{C}((\hbar)))$$

along with

$$\mu eu(\mathcal{F}) \in H^{2n}_{Z'}(T^*X, \mathbb{C})$$

Theorem (a version of Schapira-Schneiders)

$$\boxed{\chi\left(\mathbb{R}\Gamma(DR(\mathcal{M}^\bullet) \otimes_{\mathbb{C}} \mathcal{F})\right) = \int_{T^*X} \mu eu^\hbar(\mathcal{M}) \cup \mu eu(\mathcal{F})}$$

Theorem (Bressler-Nest-T)

$$\mu eu^\hbar(\mathcal{M}) = \left[ch(\sigma\mathcal{M}) \cdot \pi^* Td(T_X)\right]_{2n}$$

To summarize:

Most probably:

(i) $C_\bullet(\mathrm{Sheaves}(X)_Z) \to H^\bullet_Z(T^*X, \mathbb{DC}_\bullet(\mathbb{C}))$

$(CC_\bullet^-)$ $\qquad$ $(CC_\bullet^-)$

$$\mathrm{ch}(\mathcal{F}) \longmapsto \mu eu(\mathcal{F})$$

True:

(ii) $C_\bullet(\mathcal{D}_X\text{-mod}_{Z'}) \longrightarrow H^\bullet_{Z'}(T^*X, \mathbb{DC}_\bullet(\mathbb{C}((\hbar))))$,

$(CC_\bullet^-)$ $\qquad$ $(CC_\bullet^-)$

$$\mathrm{ch}(\mathcal{M}) \longmapsto \mu eu^\hbar(\mathcal{M})$$

$\|$ almost for sure

$\mu eu(\mathcal{M})$ of Sch-Schr

(iii) $C_\bullet(F\mathcal{D}_X\text{-mod}_{Z'}) \xrightarrow{\sigma} C^\bullet(\mathcal{O}_{T^*X}\text{-mod}_{Z'})$

$CC_\bullet^-$

$$\downarrow \mathrm{ch}$$

$$H^\bullet_{Z'}(T^*X)$$

1. $\chi(\mathcal{DR}^\bullet(\mathcal{M}) \otimes \mathcal{F}) = \int \mu eu(\mathcal{F})\, \mu eu^\hbar(\mathcal{M})$

2. $\mu eu^\hbar(\mathcal{M}) = \mathrm{ch}(\sigma(\mathcal{M})) \cdot \mathrm{Td}(T_X)_{2n}$

In K-theory:

Most probably:

① $K(\text{Sheaves}(X)_Z) \to C^{\bullet}_{Z}(T^*X, \mathbb{D}K(\mathbb{C}))$

(Beilinson's construction)

② $K(\mathcal{D}_X\text{-mod}_{Z'}) \underset{?}{\longrightarrow} C^{\bullet}_{Z'}(T^*X, \mathbb{D}K(\mathbb{C}))$

③ $K(F\mathcal{D}_X\text{-mod}_{Z'}) \xrightarrow{\sigma} K(\mathcal{O}_{T^*X}\text{-mod}_{Z'})$

$\updownarrow$?? what is the relation?

$C^{\bullet}_{Z'}(T^*X, \mathbb{D}K(\mathbb{C}))$

Questions:

- Is there $\langle , \rangle : C^{\bullet}_{Z}(T^*X, \mathbb{D}K(\mathbb{C})) \times C^{\bullet}_{Z'}(T^*X, \mathbb{D}K(\mathbb{C})) \to K(\mathbb{C})$?

- Is $\det(\mathcal{DR}^{\bullet}(\mathcal{M}) \otimes \mathcal{F}) = \langle ①(\mathcal{F}), ②(\mathcal{F}) \rangle$?

Some arguments in favor of

(2): $K(\mathcal{D}_X\text{-mod}_Z) \to C^{\cdot}_{Z'}(T^*X, \mathbb{D}K(\mathbb{C}))$.

1) A sketch of the construction

2) Secondary characteristic classes

3) Speculations about $e^{f/i\hbar}$

≡

1) Reminder about μeu of Kashiwara-Schapira.

The canonical trace:

$$(\star) \qquad \boxed{R\Gamma_c\left(\mathcal{D}_X \overset{\mathbb{L}}{\underset{\mathcal{D}_X \otimes \mathcal{D}_X^{op}}{\otimes}} \mathcal{D}_X\right) \to \mathbb{C}}$$

morphism of presheaves. Verdier dual:

$$C_{\cdot}(\mathcal{D}_X) = \mathcal{D}_X \overset{\mathbb{L}}{\underset{\mathcal{D}_X \otimes \mathcal{D}_X^{op}}{\otimes}} \mathcal{D}_X \to \mathbb{D}\mathbb{C} = \mathbb{D}C_{\cdot}(\mathbb{C})$$

Explanation / motivation:

1) A trace on an algebra A:

$$\begin{array}{c} A \overset{\mathbb{L}}{\otimes}_{A\otimes A^{op}} A \longrightarrow A/[A,A] \to \mathbb{C} \\ \qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\quad A \otimes_{A\otimes A^{op}} A \end{array}$$

2) A trace density on a sheaf of algebras: can integrate comp. supp. functions;

$$\Gamma_c(\mathcal{A}) \longrightarrow \mathbb{C}$$

Natural generalization:

$$R\Gamma_c\left(\mathcal{A} \overset{\mathbb{L}}{\otimes}_{\mathcal{A}\otimes\mathcal{A}^{op}} \mathcal{A}\right) \to \mathbb{C}$$

(at the level of explicit complexes:

$$C^\bullet(X, A \overset{\mathbb{L}}{\underset{A\otimes A^{op}}{\otimes}} A):$$

$$\begin{array}{l}
\prod_{U,V,W} \Gamma(U\cap V\cap W, \longrightarrow) \\
\uparrow\ \uparrow\ \uparrow \\
\prod_{U,V} \Gamma(U\cap V, A \leftarrow A\otimes A \leftarrow \cdots\cdots) \\
\uparrow\ \uparrow \\
\prod_{U} \Gamma(U, A \leftarrow A\otimes A \leftarrow A^{\otimes 3} \leftarrow \cdots) \\
\qquad\qquad [a_0, a_1] \leftarrow\!\shortmid a_0\otimes a_1
\end{array}$$

Hochschild-Čech bicomplex

$$R\Gamma_c(X, A \overset{\mathbb{L}}{\underset{A\otimes A^{op}}{\otimes}} A)$$

$$\| $$

$$\mathrm{Cone}\left(C^\bullet(X, A \overset{\mathbb{L}}{\underline{\otimes}} A) \to \varinjlim_{K\subset X} C^\bullet(X, A \overset{\mathbb{L}}{\underline{\otimes}} A)\right)$$

Diff. ops as kernels

Funct. anal. : all operators ~

kernels $k(x,y)\,dy$

$dy = dy_1 \ldots dy_n$

$k(x,y)$ - distribution.

Dif. ops : k supported at Δ = diagonal

$A = \sum P_n(x)\,\partial_x^n$ $\quad k(x,y) = \sum P_n(x)\,\delta^{(n)}(x-y)$

Alg/Holomorphic analysis :

$$\mathcal{D}_X \longrightarrow \mathcal{H}^n_\Delta(\mathcal{O}_X \boxtimes \Omega^n_X)$$

$$U \longmapsto H^n_\Delta(U, \mathcal{O}_U \boxtimes \Omega^n_U)$$

(Sato)

(Local computation :

$$\text{cone}\left[\Gamma(U\times U, \mathcal{O}\boxtimes\Omega^n) \to \check{C}^\bullet(\mathcal{U}; \mathcal{O}\boxtimes\Omega^n)\right]$$

$\mathcal{U}$ = cover of $U\times U - \Delta$ by $\{z_i - z_i' \neq 0\}$

Cohomology:

on $U_1 \cap \dots \cap U_n$, $f(z_1, \dots, z_n) \prod_{i=1}^{n} (z_i - z_i')^{-1} \times$

$$\times (z_1 - z_1')^{-m_1} \dots (z_n - z_n')^{-m_n}$$

corresponds to

$$f(z_1, \dots, z_n)\, \partial_{z_1}^{m_1} \dots \partial_{z_n}^{m_n}$$

$$\equiv$$

$$\mathcal{D}_X \overset{\mathbb{L}}{\underset{\mathcal{D}_X \otimes \mathcal{D}_X^{op}}{\otimes}} [\mathcal{O}_X \boxtimes \Omega_X^n]$$

$\big\downarrow$ trace density

$$i_{\Delta *}\left(\Omega_X^0 \to \dots \to \underset{\deg = -1}{\Omega_X^{n-1}} \to \underset{\deg = 0}{\Omega_X^n}\right)$$

$$k(z, z')\, dz' \mapsto k(z, z)\, dz$$

extends to

Apply $R\Gamma_c(\mathcal{H}_\Delta^n(\cdot))$:

$$R\Gamma_c\left(\mathcal{D} \overset{\mathbb{L}}{\underset{\mathcal{D} \otimes \mathcal{D}}{\otimes}} \mathcal{D}\right) \xrightarrow[tr]{} \mathbb{C}$$

Question: can

$$\mathrm{tr}: R\Gamma_c\left(\mathcal{D}_X \overset{L}{\underset{\mathcal{D}_X \otimes \mathcal{D}_X}{\otimes}} \mathcal{D}_X\right) \to \mathbb{C}$$

be categorified, i.e. does it come from a DG functor to which $K(?)$ can be applied as well?

Perhaps:

$\mathcal{D}_X$-mod w/compact support

$$= \mathcal{D}_X\text{-mod}_c = \left\{ (\mathcal{M}, \varphi) : \mathcal{M} \xrightarrow[\sim]{\varphi} 0 \text{ on } X \setminus K \right\}$$

$$\begin{tikzcd} K(\mathcal{D}_X\text{-mod}_c) \arrow[r, "\substack{\text{descent}\\ \text{map}}"'] \arrow[d, "\omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} ?"] & C_c^\bullet(X, K(\mathcal{D}\text{-mod})) \\ K(\mathbb{C}\text{-mod}) & \end{tikzcd}$$

$\boxed{\text{In the spirit of Kash-Sch}}$

a) Solutions of $\mathcal{M} \in \mathcal{D}\text{-mod}_c$ seem to be PERFECT? b) If the descent map is an equivalence, at least for a ball U, then Ok.

2) Secondary classes

Karoubi regulator.

For a Fréchet algebra A:

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d] & K^{top}(A) \arrow[d] \\ |CC_\cdot^-(A)| \arrow[r] & |CC_\cdot^{per}(A)| \end{tikzcd}$$

$|\cdot|$ is the Dold-Kan construction. The tensor product in the definition of $CC_\cdot^{per}$ is $\hat{\otimes}$.

We get

$$K(A) \longrightarrow |CC_\cdot^-(A)| \underset{|CC_\cdot^{per}(A)|}{\overset{h}{\times}} K^{top}(A)$$

Case $A = \mathbb{C}$:

$$\mathrm{reg}: \; K_{2n-1}(\mathbb{C}) \longrightarrow \mathbb{C}/\mathbb{Z}$$

Case $A = \mathcal{O}_U^\hbar$, $U \subset T^*X$:

$$K^{top}(\mathcal{O}_U^\hbar) \xleftarrow{\sim} K^{top}(\mathbb{C})$$

$$CC_\cdot^-(\mathcal{O}_U^\hbar[\hbar^{-1}]) \simeq \bigoplus_{j \geq 0} \mathbb{C}((\hbar))[2n-2j]$$

$$CC_\cdot^{per}(\mathcal{O}_V^\hbar) \xrightarrow{\sim} \bigoplus_{j\in\mathbb{Z}} \mathbb{C}((\hbar))[2j]$$

Compose Karoubi's regulator with the descent map:

$$\begin{array}{c}
K(F\mathcal{D}_X\text{-mod}_Z) \longrightarrow K(\mathcal{O}^\hbar_{T^*X}\text{-mod}) \\
\downarrow \\
C^\cdot_Z(T^*X, K(\mathcal{O}^\hbar_{T^*X})) \\
\downarrow \\
C^\cdot_Z(T^*X, |CC_\cdot^-(\mathcal{O}^\hbar)| \times_{|CC_\cdot^{per}(\mathcal{O}^\hbar)|} K^{top}(\mathbb{C}))
\end{array}$$

$$K_0(F\mathcal{D}_X\text{-mod}_Z) \to \bigoplus_{j=0}^{2n} H_Z^{2j}(T^*X, \mathbb{Z}(j))$$

$$\oplus \bigoplus H_Z^{2n+2j+1}\left(T^*X, \mathbb{C}((\hbar))/\mathbb{Q}(n+j+1)\right)$$

$\mathbb{Q}$ instead of $\mathbb{Z}$ appeass because the map $K^{top}(\mathbb{C}) \to CC_\cdot^{per}(\mathcal{O}^\hbar_{T^*X})$

induces the map

$$\bigoplus_{j\geq 0} \mathbb{Z}(j)[2j] \longrightarrow \bigoplus_{j\in\mathbb{Z}} \mathbb{C}((\hbar))[2j]$$

which is the standard inclusion followed by multiplication by

$$\sum u^{-p}\cdot Td(T_X)^{2p}$$

This introduces the denominators.

<u>Thm</u> (Bressler, T)
If $\mathcal{M}$ is a local system, the image of $\mathcal{M}$ in

$$\bigoplus H_X^{2n+2j+1}(T^*X, \mathbb{C}((\hbar))/\mathbb{Q}(n+j+1))$$

$$\downarrow$$

$$\bigoplus H^{2j+1}(X, \mathbb{C}((\hbar))/\mathbb{Q}(j+1))$$

is given by the Čech cocycle

$$U_{i_0}\cap\ldots\cap U_{i_{2j+1}} \mapsto reg(g_{i_0 i_1}|\ldots|g_{i_{2j}i_{2j+1}})$$

induced by the map

$$\mathbb{Z}BGL(\mathbb{C}) \to BGL^+(\mathbb{C}) \xrightarrow{\text{reg}} |CC^-_\cdot(\mathbb{C})| \underset{|CC^{per}_\cdot(\mathbb{C})|}{\times} K^{top}(\mathbb{C}$$

<u>Conjecture</u> The above map is induced by

$$\begin{array}{c} K(\mathcal{D}_X\text{-mod}_Z) \to C^\cdot_Z(T^*X, \mathbb{D}K(\mathbb{C})) \\ \downarrow \mathbb{D}\,\text{reg} \\ C^\cdot_Z(T^*X, \bigoplus (\mathbb{C}/\mathbb{Z}(j+n))[2n+j+1]) \end{array}$$

3) Speculations about $e^{f/i\hbar}$.

The Hochschild and cyclic homology of $\mathbb{O}^\hbar_U$ is rather complicated. It becomes simpler if we invert $\hbar$:

$$HH(\mathbb{O}^\hbar_{T^*X}[\hbar^{-1}]) \simeq \mathbb{D}\,HH(\mathbb{C}((\hbar)))$$

It seems that to compare $K(\mathbb{O}^\hbar_{T^*X})$ to $\mathbb{D}K(k)$ may be easier if one allows elements $e^{f/i\hbar}$.

Formally speaking:

$$e^{x} \neq 0 \quad \text{in} \quad K_1(\mathbb{O}^{\hbar}_{U}) \qquad \underline{(n=1)}$$

but

$$e^{\frac{\xi^2}{2i\hbar}} \cdot e^{x} \cdot e^{-\frac{\xi^2}{2i\hbar}} = e^{x+\xi}$$

so $e^{x}, e^{\xi} = 0$ in the new K_1.

On the other hand:

$e^{2\pi x}$ and $e^{\xi/i\hbar}$ commute, so,

formally,

$$\{e^{2\pi x}, e^{\frac{\xi}{i\hbar}}\}_{\text{Milnor}} \in K_2$$

Again, formally, $K_2 \to HH_2$ sends this element to the generator.

Appendix

Hochschild complex of a DG category $\mathcal{A}$:

$$C_p(\mathcal{A}) = \bigoplus_{i_0,\dots,i_p \in Ob(\mathcal{A})} \mathcal{A}(i_0,i_1) \otimes \bar{\mathcal{A}}(i_1,i_2) \otimes \dots \otimes \bar{\mathcal{A}}(i_p,i_0)$$

$$\bar{\mathcal{A}}(i,j) = \mathcal{A}(i,j),\ i \neq j;\quad \mathcal{A}(i,i)/k \cdot 1_i,\ i=j$$

$$b : C_p(\mathcal{A}) \to C_{p-1}(\mathcal{A})$$

$$b(a_0 \otimes \dots \otimes a_p) = \sum_{i=0}^{p-1} \pm a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_p$$

$$\pm a_p a_0 \otimes a_1 \otimes \dots \otimes a_{p-1}$$

$$d(a_0 \otimes \dots \otimes a_p) = \sum_{i=0}^{p} \pm a_0 \otimes \dots \otimes da_i \otimes \dots \otimes a_p$$

Total differential: $b+d$

Total grading: $-p+$ (grading of $\mathcal{A}$)

The cyclic differential:

$$B(a_0 \otimes a_1 \otimes \dots \otimes a_p) = \sum_{j=0}^{p} \pm 1 \otimes a_{j+1} \otimes \dots \otimes a_0 \otimes \dots \otimes a_j$$

The negative cyclic complex:

$$CC^-_\bullet(A) = C_\bullet(A)[[u]] \qquad |u| = 2$$

$$b + d + uB$$

The periodic cyclic complex:

$$CC^{per}_\bullet(A) = C_\bullet(A)((u))$$

$$b + d + uB$$

K-theory of a DG category $\mathcal{A}$

DG module: complexes $\mathcal{M}(i)$, $i \in Ob(\mathcal{A})$

$\mathcal{A}(i,j) \otimes \mathcal{M}(j) \to \mathcal{M}(i)$ subject to...

Free DG module: $V(i)$, $i \in Ob(\mathcal{A})$ — fin. complexes; $V(i) = 0$ a.e.

$$(\mathcal{A} \otimes V)(i) = \bigoplus_{j \in Ob(\mathcal{A})} \mathcal{A}(i,j) \otimes V(j)$$

Semifree DG-module: obtained from free by a finite number of extensions.

$Mod(\mathcal{A}) := \{$Retracts of semifree DG-mods$\}$

Weak equivalence: Morphism of DG functors $\mathcal{M}(i) \to \mathcal{N}(i)$

quasi-isomorphisms

Exact sequences in $Mod(\mathcal{A})$:

$$\mathcal{M}_1 \rightarrowtail \mathcal{M}_2 \twoheadrightarrow \mathcal{M}_3$$

Waldhausen S-construction

$$K_0(\mathcal{M}od(A)) = \bigoplus_{M \in Ob(\mathcal{M}od(A))} \mathbb{Z}\cdot[M]/\sim$$

① $[M] \sim [M']$ if $M \xrightarrow{w} M'$ weak eq.

② $[M_2] \sim [M_1] + [M_{12}]$ for

$$M_1 \rightarrowtail M_2 \twoheadrightarrow M_{12}$$

How to "resolve" both?

① Pass to the nerve of the category of weak equivalences

② Pass to the S-construction; apply nerve ① to that. $\mathcal{C} := \mathcal{M}od(A)$:

$S_0\mathcal{C} = *$; $S_1\mathcal{C} = \mathcal{C}$;

roughly speaking,

$$S_n\mathcal{C} = \{ M_1 \rightarrowtail M_2 \rightarrowtail \dots \rightarrowtail M_n \}$$

More precisely:

$$\mathrm{Ob}\, S_1\mathcal{C} = \{ M_1 \}; \quad \mathrm{Ob}\, S_2\mathcal{C} = \{ M_1 \rightarrowtail M_2 \twoheadrightarrow M_{12} \}$$

...

$$\mathrm{Ob}\, S_n\mathcal{C} = \left\{ \begin{tikzcd}[column sep=small, row sep=small]
 & & & M_{n-1,n} \\
 & & & \vdots \arrow[u, two heads] \\
 & M_{23} \arrow[r, tail] & \arrow[r, tail] & M_{2n} \\
M_{12} \arrow[r, tail] \arrow[ur, two heads] & \cdots \arrow[r, tail] & \arrow[r, tail] & M_{1n} \arrow[u, two heads] \\
M_1 \arrow[r, tail] & M_2 \arrow[r, tail] \arrow[ul, two heads] & \cdots \arrow[r, tail] & M_n \arrow[u, two heads]
\end{tikzcd} \right\}$$

(filtrations with chosen subquotients).

$$\begin{tikzcd}
M_1 \arrow[r, tail] \arrow[d, "d_0"'] & M_2 \arrow[r, two heads] \arrow[d, "d_1"] & M_{12} \arrow[d, "d_2"] \\
M_1 & M_2 & M_{12}
\end{tikzcd}$$

and analogously for higher n and for s_i.

$S_\bullet\mathcal{C}$ – simplicial category.

$wS_\bullet\mathcal{C}$ – the same simplicial category, with weak equivalences only.

$$\boxed{K(A) = \Omega \left| \text{Nerve } wS.\mathcal{C} \right|}$$

where $\mathcal{C} = \mathcal{M}od(A)$

The Chern character

I think this is known/true:

$$\boxed{K(A) \xrightarrow{ch} |CC_\bullet^-(A)|}$$

Not quite obvious.