Arinkin 1

Singular support of coherent sheaves

$X = \operatorname{Spec} R/(f_1, \dots, f_n)$

$\cap$ Zero locus $\quad f_1 = \dots = f_n = 0$

$U =$ smooth $\quad /\mathbb{C}$

$\|$

$\operatorname{Spec} R$

Either
- assume $\operatorname{codim} X = n$
- any f_i's $\quad X$ dg

(one example: $f_1 = \dots = f_n = 0$)

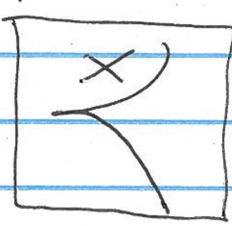
Key observation (Gulliksen '74)

For any $F = A$-module

$\xi_1, \dots, \xi_n \in \operatorname{Ext}^2_A(F, F)$

Example $\quad A = R/(f)$; $\mathcal{F}$ a qcoh sheaf on $X \hookrightarrow U$

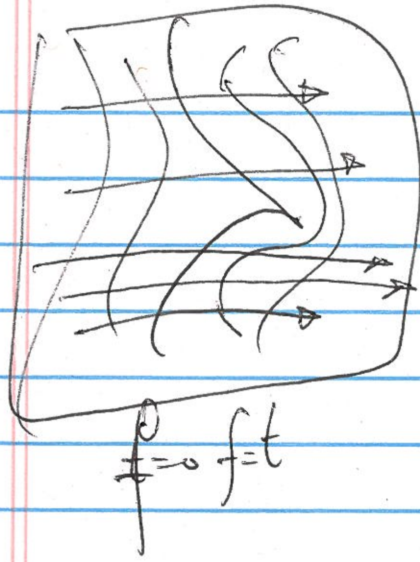
$$i^* i_* F = F$$



U

$$L_k i^* i_* F = \begin{cases} F, & k = 0, 1 \\ 0, & k > 1 \end{cases}$$

$$F[1] \to L i^* i_* F \to F \xrightarrow{\xi} F[2]$$



non Trivial

(ξ_i come from $H^2(A, A)$; correspond to an n-parameter family of deformations $f_i = t_i$

For the same reason, they <u>commute</u> in $\mathrm{Ext}^{\bullet}_A(F, F)$

<u>Theorem</u> (Gulliksen) If $\mathcal{F} \in D^b_{coh}(X)$ then $\mathrm{gr}\,\mathrm{End}(\mathcal{F})$ is a fin. gen. module / $\cancel{A}[\xi_1, \dots, \xi_n]$

$$\mathrm{gr}\,\mathrm{End}(F) = \bigoplus_k \mathrm{Ext}^k(F, F) \leftarrow \cancel{A}[\xi_1, \dots, \xi_n]$$

<u>Definition</u> $\mathrm{Sing\,Supp}(F) = \mathrm{supp}_{A[\xi_1, \dots, \xi_n]} \mathrm{gr}\,\mathrm{End}(F)$

$$\cap$$

$$\underset{(\xi_1, \dots, \xi_n)}{X \times \mathbb{A}^n} = \mathrm{Spec}\, A[\xi_1, \dots, \xi_n]$$

<u>Properties</u>

1) $\mathrm{Supp}_A\, \mathrm{gr}\, \mathrm{End}(F) = \mathrm{supp}\, F$

2) projection of $\mathrm{sing\,supp}(F)$ onto X is $\mathrm{supp}(F)$

3) $\mathrm{Sing\,Supp}(F)$ is conical ($\mathrm{gr\,End}$ is <u>graded</u>)

3) If F is perfect, $\mathrm{Ext}^k(F,F)$ is bounded
$= 0$, $k \gg 0$

$$\mathrm{Sing\,Supp}(F) = \mathrm{Supp}\,F \times \{0\}$$
$\cup$

This can be reversed: $\mathrm{Sing\,Supp} \Rightarrow \mathcal{F}$ perfect.

$=$

$\mathcal{F} \in D(X)$ - qcoh Derived cat.

Its graded center: $\mathrm{gr}\,Z_{D(X)} = \bigoplus_k \mathrm{Hom}\left(\mathrm{Id}_{D(X)}, \mathrm{Id}_{D(X)}[k]\right)$

$$\begin{tikzcd} A[\xi_1, \dots, \xi_n] \arrow[r] \arrow[dr] & \mathrm{gr}\,Z_{D(X)} \arrow[r] & \mathrm{gr}\,\mathrm{End}(F,F) \\ & HH^*(X) \arrow[u] & \end{tikzcd}$$

Hochschild cohomology

By this naturality: $\{x\} \xhookrightarrow{i_x} X$
$\quad \Cap \qquad\quad \Cap$
$\quad X \qquad\quad \mathcal{F}$

$A[\xi_1, \dots, \xi_n]$ acts on $H^\bullet(i_x^! F) = \mathrm{Ext}^\bullet(\mathbb{C}_x, \mathcal{F})$

Claim $\mathrm{supp}\, H^\bullet(i_x^! \mathcal{F}) = (\{x\} \times \mathbb{C}^n) \cap \mathrm{Sing\,Supp}(\mathcal{F})$

Example X is a "dg-point": $U = pt$; $f_1 = \dots = f_n = 0$

$X = \operatorname{Spec} A$ $\qquad A = \mathbb{C}[\eta_1, \dots, \eta_n]$

Consider $i_x^!(F)$ $\qquad d\eta_i = 0 \qquad \deg \eta_i = -1$

$$D^b_{coh}(X) \xrightarrow{i_x^!} D_{coh}(\mathbb{C}[\xi_1, \dots, \xi_n])$$

$|\eta_i| = -1 \qquad |\xi_i| = 2$

Sing Supp ——— Usual supp

Koszul transform

(plays a role similar to the Fourier transform)

<u>Where does Sing Supp actually lie?</u>

Consider T^*X as a <u>complex</u>

$$(T^*U)|_X \xleftarrow{df_1, \dots, df_n} \mathcal{O}_X^n$$

$$\sum a_i df_i \longleftarrow\!\shortmid (a_1, \dots, a_n)$$

Consider "$H^{-1}(T^*X)$" $\qquad X \times \mathbb{A}^n$

$$\{(x, a_1, \dots, a_n) \in X \times \mathbb{A}^n : \sum a_i df_i(x) = 0\} = H^{-1}(T^*X) \subset X \times \mathbb{A}^n$$

$\|$

Spec Sym $TX[1]$ ...

<u>Claim</u> For any $F \in D^b_{coh}(X)$

$$\mathrm{Sing\,Supp}(F) \subset \text{"}H^{-1}(T^*X)\text{"}$$

any closed conical subset appears as Sing Supp of some F

As a subset of "$H^{-1}(T^*X)$", Sing Supp(F) depends only on X and on F.