Arinkin 2

Summary:

$$X = \{ f_1 = \dots = f_n = 0 \}$$
$$\cap$$
U – smooth affine

$$\begin{array}{c} H^{-1}T^*X \\ \Big\downarrow \text{vector spaces varying dim} \\ X \end{array} \qquad \begin{array}{c} \mathcal{F} \in D^b_{coh}(X) \\ \Big\downarrow \\ \mathrm{Sing\,Supp}(\mathcal{F}) \end{array}$$

Def (Avramov–Buchweitz, Benson–Iyengar–Krause)

Rem $\mathrm{Sing\,Supp}(\mathcal{F}) \cap$ zero section $= \mathrm{supp}(\mathcal{F})$

What is $\mathrm{Sing\,Supp}(\mathcal{F})$ – zero section?

$$[F] \in D^b_{coh}(X)/\mathrm{Perf}(X) = D_{sing}(X)$$

singularity category of X

Thm (Orlov)

~~Set $Y = \{ (u, b_1, \dots, b_n) \in U \times \mathbb{P}^1 \}$~~

Set $Y = \{ (u, b_1, \dots, b_n) \in U \times \mathbb{P}^{n-1} \mid f_1(u)b_1 + \dots + f_n(u)b_n = 0 \}$

$$D_{sing}(X) \simeq D_{sing}(Y)$$

Ex Y - hypersurface, its sing locus $=$

$$H^{-1}(T^*X) - \text{zero section} \Big/ \mathbb{G}_m$$

Claim $\mathcal{F} \in D^b_{coh}(X)$, $\mathrm{Sing\,Supp}(\mathcal{F})$ -

$$- \text{zero section} \Big/ \mathbb{G}_m \subset Y$$

is the "support" of $\Phi([F])$, i.e. the smallest closed subset s.t.

$\Phi([F])$ is perfect on the complement.

(2) Sing Supp is local in Zariski topology, so it makes sense if X is l.c. (not necess. affine) (also smooth-local, so X could be a stack)

So can look at $D^b_{coh}(X)$ makes sense to enlarge it.

Ex Stack of local systems on a curve

$$\begin{array}{c} \mathcal{L}oc\,Sys \\ \uparrow \\ \mathcal{L}oc\,Sys_x \end{array} \quad \text{framed at one point}$$

$$\mathrm{Coh}(\mathcal{L}oc\,Sys) = \underline{\mathcal{L}oc\,Sys_x \hookrightarrow} \mathcal{L}oc\,Sys_x^{reg\ sing}$$

Equation: first residue vanishes ...

$$\begin{tikzcd} \mathcal{L}oc\,Sys & H^{-1}T^*\mathcal{L}oc\,Sys \arrow[l] \\ \mathcal{L}oc\,Sys_x \arrow[u, "G"] \arrow[r] & H^{-1}T^*\mathcal{L}oc\,Sys_x \arrow[u] \end{tikzcd}$$

$$H^{-1}T^*\mathcal{L}oc\,Sys = H^{-1}T^*\mathcal{L}oc\,Sys_x / G$$

Exercise: (G reductive)

$$H^{-1}T^*\mathcal{L}oc\,Sys = \left\{ L \in \mathcal{L}oc\,Sys,\ \underset{\substack{\text{horizontal section} \\ \text{of } ad_L}}{A} - \text{infinitesimal symmetry} \right\}$$

$$\mathrm{Ind\,Coh}(X) \overset{\equiv}{=} QC^!(X)$$

Def (Krause) $\mathrm{Ind\,Coh}(X)$ = homotopy cat of complexes of q coh sheaves on X

$\mathrm{Ind\,Coh}(X)$ = triang cat with ∞ $\oplus$ compactly generated by $D^b_{coh}(X)$!

$$\begin{tikzcd} \mathrm{Ind\,Coh}(X) \\ D^b_{coh}(X) \arrow[u, hook, "\substack{\text{inj} \\ \text{resolutions}}"'] \end{tikzcd}$$

Other defs View $D^b_{coh}(X)$ as a dg category. Set $\mathrm{Ind\,Coh}(X) =$
= Ind its completion

• (Positselski) "Coderived category"

$$QC^*(X) = D(X) = \mathrm{Ind}(\mathrm{Perf}(X))$$
$$= \mathrm{Ind\,Coh}_{\text{zero section}}(X)$$

$$QC^!(X) = \mathrm{Ind\,Coh}(X) = \mathrm{Ind}(\mathrm{Coh}(X)) =$$
$$= \mathrm{Ind\,Coh}_{H^{-1}(T^*X)}(X)$$

There is a notion of $\mathrm{Sing\,Supp}(\mathcal{F})$ for

$$\mathcal{F} \in \mathrm{Ind\,Coh}(X) \qquad \hat{H}^{-1}(T^*X)$$

For any conical $Y \subset H^{-1}(T^*X)$, take

$$\mathrm{Ind\,Coh}_Y(X) = \{ \mathcal{F} \in \mathrm{Ind\,Coh}(X) \mid$$
$$\mathrm{Sing\,Supp}(\mathcal{F}) \subset Y \}$$

<u>Define</u> $\mathrm{Sing\,Supp}(\mathcal{F})$ for $\mathcal{F} \in \mathrm{Ind\,Coh}(X)$ by

sampling: $\mathrm{Ext}^\bullet_{\mathrm{Ind\,Coh}}(\mathcal{G}, \mathcal{F})$

$$\mathcal{G} \in D^b_{coh}(X)$$

$$\circlearrowright$$

$$\xi_i$$

Sing Supp has nice functoriality

$$X_1 \xrightarrow{f} X_2 \qquad \begin{tikzcd} \mathrm{IndCoh}(X_1) \arrow[d, shift right, "f_*"'] \\ \mathrm{IndCoh}(X_2) \arrow[u, shift right, "f^!"'] \end{tikzcd}$$

$$\begin{tikzcd} H^{-1}T^*X_1 & & H^{-1}T^*X_2 \\ & \left(H^{-1}T^*X_2\right) \underset{X_2}{\times} X_1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

the usual functoriality

$$f_* \, \mathrm{IndCoh}_{Y_1}(X_1) \subset \mathrm{IndCoh}_{Y_2}(X_2)$$

...