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Donaldson-Thomas type invariants via microlocal geometry

$DT(Y) \qquad \chi(X, \nu_X)$

CY3

moduli space

1. DT invariants:

Y: CY 3-fold (proj n.s scheme$/\mathbb{C}$ of dim 3)

w/ $\Omega^3_Y = \omega_Y \cong \mathcal{O}_Y$

e.g. quintic hypersurface in $\mathbb{P}^4$ e.g. $x_0^5 + \dots + x_4^5 = 0$

X - a moduli space of coherent sheaves on Y with trivialized determinant (& fixed invariants)

ex $X = \mathrm{Hilb}^n(Y) = \{$rank 1 torsion free sheaves, i.e. ideal sheaves s.t. $\dim_{\mathbb{C}} \mathcal{O}/I = n\}$

$X = \{ \dots \mathcal{O}/I = \mathcal{O}_Z$ structure sheaf of a "curve" i.e. 1 dim subscheme $\}$

$X = \left\{ \begin{array}{l} \text{stable sheaves of given Chern} \\ \text{classes} \end{array} \right\}$

X - stack

Donaldson-Thomas define $\#^{vir} X$ = "virtual # of points of X"

e.g. X = moduli space of lines on quintic $\mathbb{CP}^4$
= discrete set of points (conics)

$\#^{vir} X = \# X = 2875 \qquad (609250)$

In general $\dim X > 0$ highly singular (& non-reduced)

<u>DT invariants</u> defined using deformation thy and intersection theory

$T_X \Big|_{[E]} = \quad Ext^1(E,E) = H^1(Y, End\, E)$ if E bundle — infin defs

$ob \Big|_{[E]} = Ext^2(E,E)$ - obstruction space $\supset$

$\supset$ obstructions to smoothness X at $[E]$

Suppose X smooth. ~~and~~ THEN $\dim Ext^2(E,E)$ is constant as $[E] \in X$ varies then

$\exists$ VB $\mathcal{E}/X$ w/ fiber over $\mathrm{Ext}^2(E,E)$
and if X is proper:

$$\#^{vir}(X) = \int_{[X]} c_{top}\, \mathcal{E}$$

need $\mathrm{rk}\,\mathcal{E} = \dim X$

$$\mathrm{Ext}^2(E,E) = \mathrm{Ext}^1(E,E)^\vee$$

Serre duality on 3D CY

In fact $\mathcal{E}^\vee = T_X \Rightarrow \mathcal{E} = \Omega_X$
and

$$\#^{univ} = \int_{[X]} c_{top}(\Omega_X) =$$

$$= (-1)^{\dim X} \int_{[X]} c_{top}\, T_X = (-1)^{\dim X} \chi(X)$$

(Gauss-Bonnet)

topological Euler char

Generalize this to singular X.

2. Symmetric obstruction theories

Example (Toy model) M: smooth scheme$/\mathbb{C}$

$$f : M \to \mathbb{C} \quad \text{reg fn}$$

$$X = \mathrm{Crit}(f) \subset M \quad \text{subscheme} = Z(df)$$

defined:

$$\begin{tikzcd} T_M \arrow[rr, "df"] \arrow[dr] & & \mathcal{O}_M \\ & I \arrow[ur] & \end{tikzcd}$$

ideal sheaf of $X \subset M$

$$\begin{tikzcd} \left[T_M|_X \arrow[r, "H(f)"] \arrow[d] & \Omega_M|_X \right] \arrow[d, equal] & = E^\bullet \in D_{[-1,0]}(\mathcal{O}_X) \\ \left[I/I^2 \arrow[r, "d"] & \Omega_M|_X \right] & = \tau_{\geq -1} L_X \end{tikzcd}$$

Conormal sheaf

the truncation of the cotangent complex

Get a <u>perfect obstruction theory</u>

$$E^\bullet \to \tau_{\geq -1} L_X$$

$$\begin{cases} H^0 \xrightarrow{\sim} H^0 \\ H^{-1} \twoheadrightarrow H^{-1} \end{cases}$$

which is <u>symmetric</u> i.e. $E^\bullet \xrightarrow{\sim} E^\vee[1]$

by symmetry of $H(f)$

Why an obstruction theory?

$$\begin{tikzcd}
T_M|_X \arrow[r, "H(f)"] \arrow[d] & \Omega_M|_X \arrow[r] \arrow[d, equal] & \Omega_X \arrow[d, equal] & \\
I/I^2 \arrow[r] & \Omega_M|_X \arrow[r] & \Omega_X \arrow[r] & 0
\end{tikzcd}$$

<u>dualize</u>:

$$\begin{tikzcd}
0 \arrow[r] & T_X \arrow[r] & {[T_M|_X \to \Omega_M|_X]} \arrow[r] & \mathrm{ob} \\
0 \arrow[r] & T_X \arrow[r] & {[T_M|_X \to N_{X/M}]} \arrow[r] & \mathrm{cok} \arrow[u]
\end{tikzcd}$$

actual obstructions to smoothness

$$\mathrm{Spec}\left(\oplus I^n/I^{n+1}\right) \to C_{X/M} \to CV$$

Curvilinear obstructions

$$[X]^{vir} = 0^!_{\Omega_M}[\Gamma_{df}]$$

$$\in A_0(X)$$

Chow group of cycles modulo rat'l equiv

$$\begin{tikzcd}
M \arrow[r, hook, "\Gamma_{df}"] & \Omega_M \\
X \arrow[u, hook] \arrow[r, hook] & M \arrow[u, hook, "0"']
\end{tikzcd}$$

If X cpt:

$$\#^{vir} X = \deg [X]^{vir} = \int_{[X]^{vir}} 1 \in \mathbb{Z}$$

$$\|$$

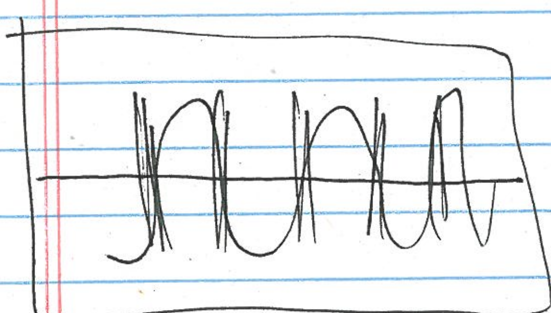
$$I_{\Omega_M}(0, df)$$

Def. to normal cone:

$$t\,df \qquad t \in \mathbb{C} \qquad |t| \to \infty$$

get cone inside $\Omega_M|X \hookrightarrow \Omega_M$

this _is_ the normal cone $C_{X/M}$ embedded into Ω_M by df.



Then $[X^{vir}] = 0^!_{\Omega_M|X} [C_{X/M}]$

depends only on $E \to L_X$

on the obstruction thy

Moreover, $[C_{X/M}] \subset \Omega_M$ is a conic Lagrangian cycle

So it is a characteristic cycle of a constr fn $\mu : X \to \mathbb{Z}$

which turns out to be:

$$\mu(P) = (-1)^{\dim M}\left(1 - \chi\left(\begin{matrix}\text{Milnor fibre} \\ \text{of } f \text{ at } P \in X\end{matrix}\right)\right)$$

<u>Microlocal index theorem</u> (Kashiwara)

X cpct: $\quad \#^{vir}(X) = \deg 0^!_{\Omega_{M|X}} [C_{M|X}]$

$$\| $$

$$\chi(M, \mu)$$

Each critical point gives a contribution in terms of its Milnor fibre.

<u>Global case</u> Let X be a scheme with a symmetric obstruction thy.

$$E \to \tau_{\geq -1} L_X \;;\; E \simeq E^\vee[1]$$

$E^\bullet \to L_X$ ~~gives [illegible]~~

given by $E = R\pi_* R\mathcal{H}om(\mathcal{E}, \mathcal{E})[2]$

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$E \xrightarrow{\sim} E^\vee[1]$ Serre duality on 3D CY

$$\begin{tikzcd} \mathcal{E} \arrow[dr] & & \\ & X \times Y \arrow[r, "\pi"] & X \end{tikzcd}$$

Pick $X \hookrightarrow M$ smooth

~~Pick $X \hookrightarrow M$ s.~~

$$\begin{tikzcd} {[E^{-1} \to E^0]} \arrow[d, "\varphi^{-1}"'] \arrow[rd, phantom, ""] & = E \arrow[d, "\varphi"] \\ {[I/I^2 \to \Omega_M|_X]} & = \tau_{\geq -1} L_X \end{tikzcd}$$

$$E^{-1} \xrightarrow{\varphi^{-1}} I/I^2, \quad E^0 \xrightarrow{\varphi^0} \Omega_M|_X$$

$ob = \Omega_X$ (for a symmetric obstruction theory, always so?...)

$$\begin{tikzcd} C \subset \Omega_M|_X \hookrightarrow \Omega_M \arrow[d] & \\ cV \arrow[r] & \Omega_X \end{tikzcd}$$

$$\boxed{[X]^{vir} = 0^!_{\Omega_M|_X}[C]}$$

(étale?)

<u>Fact</u>: locally in M $\exists$ <u>almost</u> closed 1-form ω on M s.t. $X = Z(\omega)$

$$d\omega\big|_{Z(\omega)} \subset \Omega^2_M\big|_{Z(\omega)} \qquad \omega = \sum f_i \, dx_i$$

almost closed: $\quad \dfrac{\partial f_i}{\partial x_j} \equiv \dfrac{\partial f_j}{\partial x_i} \ (f_1, \dots, f_n)$

<u>Fact</u>: deformation to normal cone defines Lagrangian cycle.

$$\begin{tikzcd} Z_*(X) \arrow[r, "\mathrm{Eu}", "\sim"'] \arrow[ddr, bend right, "c_0^M"'] & \mathrm{Con}(X) \arrow[r, leftrightarrow, "\mathrm{Char}", "\sim"'] \arrow[dd, "c_0^{SM}"] & \mathcal{L}_X(\Omega_M) \arrow[ddl, bend left, "0^!_{\Omega_M|X}"] \\ \pi_X \arrow[r, leftrightarrow] & \nu_X \arrow[r, leftrightarrow] & [C] \\ & A_0(X) & \end{tikzcd}$$

distinguished cycle of X

$$\pi_X = \sum_C (-1)^{\dim \pi(C)} \, \mathrm{null}(C^{\circ}) \, \pi(C)$$

X cpt: $\boxed{\#^{vir} X = \chi(X, \nu_X)}$

Now compute DT using stratification of X...