<u>Motivation</u> $X/\mathbb{F}_q$ $\mathcal{F}$ $\mathbb{Q}_\ell$-sheaf

$$R\Gamma(X_{\bar{\mathbb{F}}_q}, \mathcal{F}) \circlearrowleft Fr$$

$$\det(-Fr, R\Gamma) = \prod_{x \in X} \varepsilon_x(\mathcal{F}, \psi)$$

can replace ψ by rational 1-form

$$\psi : \mathbb{A}_F/F \to \bar{\mathbb{Q}}_\ell^\times$$

$$\psi = (\psi_x) \qquad \psi_0 : \mathbb{F}_q \to \mathbb{Q}_\ell^\times$$

fix this:

Conjectured by Deligne, proven by Laumon

Does this formula have a geometric version?

$\det R\Gamma(X, \mathcal{F})$ a line

ν, $x \overset{?}{\mapsto} \varepsilon_x(\mathcal{F}, \nu)$ of completely local origin

$$\eta : \bigotimes_{x \in X} \varepsilon_x(\mathcal{F}_x, \nu) \xrightarrow{\sim} \det R\Gamma(X, \mathcal{F})$$

It is *probable* that proper understanding of Laumon's work would lead to the above.

$X/\mathbb{C}$ $\quad M$ holonomic $\mathcal{D}$-module on X

$R\Gamma_{dR}(X, M)$ purely algebraic, makes sense over any field

A perverse sheaf on the *classical* topology. $F \doteq dR_M$

$$\rho: \quad R\Gamma_{dR}(X, M) \xrightarrow{\sim} R\Gamma(X_{cl}, F)$$
$$\| $$
$$R\Gamma_B(X, M)$$

$\det(\rho)$: if both sides have their rational structures: $\boxed{\det(\rho) \in \mathbb{C}^\times / \mathbb{Q}^\times}$

Left hand side: if $M/\mathbb{Q}$

R.H.S.: a perverse sheaf has its own rational structure (often...)

How often does one have a rational structure on F? For example for GM systems...

Example: B-function is a period of a regular connection with singularities

at $0, 1, \infty$ and holonomies $\alpha, \beta, -(\alpha+\beta)$.
$\Gamma(\alpha), \ldots$: local ε-factors.

(hol.) $\mathcal{D}$-module $M \longmapsto \begin{cases} x \longmapsto \mathcal{E}_{dR,x}(M,\nu) \\ \eta_{dR} : \bigotimes_{x \in X} \mathcal{E}_{dR,x} \xrightarrow{\sim} \det R\Gamma_{dR}(X, M) \end{cases}$

Constructible sheaf $\mathcal{F} \longmapsto \begin{cases} x \longmapsto \mathcal{E}_{B,x}(\mathcal{F},\nu) \\ \eta_B : \bigotimes_{x \in X} \mathcal{E}_{B,x} \xrightarrow{\sim} \det R\Gamma(X_{cl}, \mathcal{F}) \end{cases}$

(the second one purely topological)

$\mathcal{F} = dR_M$: $\quad \rho_x^{\varepsilon} : \mathcal{E}_{dR,x}(M,\nu) \xrightarrow{\sim} \mathcal{E}_{B,x}(\mathcal{F},\nu)$

$$\begin{tikzcd}
\det R\Gamma_{dR} \arrow[r, "\det \rho"] & \det R\Gamma_B \\
\bigotimes_{x \in X} \mathcal{E}_{dR} \arrow[u] \arrow[r, "\otimes \rho_x^{\varepsilon}"] & \bigotimes_{x \in X} \mathcal{E}_B \arrow[u]
\end{tikzcd}$$

(Deligne, notes by Laumon, site of GeomLang Uof)

Reg sing: Andersen, ?, Terasoma, Sabbah, ...
Bloch-Deligne-Esnault ...

The fact that the diagram <u>commutes</u> does not follow from the construction <u>at all</u>.

<u>Will</u> use ergodicity of some flow on the Teichmüller space.

<u>Rmk:</u> good to have in mind CFT on X.

$$\eta(v) : \overset{\exists}{\bigotimes_x} \mathcal{E}(v) \longrightarrow \det R\Gamma(\ldots)$$

Move v: $v_0 \to v_t$.

Things happen at: singularities of v
— " — " — of $\mathcal{F}$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\mu \ldots$

Singularities of v may $\begin{cases} \text{move} \\ \text{split} \end{cases}$

$$\mathcal{E}_x \xleftarrow[\theta_x]{\sim} \bigotimes \mathcal{E}_{x_t}$$

key local factorization property
like a connection, but a stronger property
the factorization line structure

Factorization lines:

X – smooth curve

$T \subset X$ – finite subscheme

$\mathrm{div}(X)$ $\mathbb{Z}$-divisors on X

$2^{|T|}$

$$\mathbb{D} \ni \{(D, c, \nu)\} : D \in \mathrm{Div}(X)$$

$$c \in 2^T \left\{ \begin{array}{l} \text{open} \\ \text{subscheme} \end{array} \right.$$

ν – a trivialization of $\omega_X(D)|_{(|D| \cup T_c)}$

Condition: $D \cap T \subset T_c$

$\mathcal{E}$: de Rham lines on $\mathbb{D}$ (graded rank 1 local system)

Constant line $\mathcal{E}(X)$

Suppose X proper, ν non-zero rational diff form

$D = -\mathrm{div}(\nu)$, $|T|$,

ν – trivialization of $\omega_X(D)$

Every rational differential form defines a point of this space.

$$\mathcal{E}(X) := \mathcal{E}_{(-\mathrm{div}(\nu),\, |T|,\, \nu)}$$

$$\boxed{\begin{array}{c}\mathcal{E} := \text{de Rham lines on } \mathbb{D} \\ (\mathbb{Z}\text{-graded!})\end{array}}$$

de Rham line = line bundle w/ flat connection

Does not depend on ν:

$\{$Rational forms $\neq 0\}$ = ∞-dim v.s. $- \{0\}$

Contractible.

All dR lines canonically isom?

=

Given a holonomic $\mathcal{D}$-module on a curve X:
canonically, a line (De Rham line on $\mathbb{D}$)

$$\eta : \mathcal{E}_{dR}(\mathcal{M})(X) \xrightarrow{\sim} \det R\Gamma_{dR}(X, \mathcal{M})$$

Factorization structure:

$(D_\alpha, c_\alpha, \nu_\alpha)$ - finite collection of disjoint pts of $\mathbb{D}$

Factorization structure on $\mathcal{E}$: a rule assigning to every such data an identification

$$\otimes \, \mathcal{E}_{(D_\alpha, c_\alpha, \nu_\alpha)} \xrightarrow{\sim} \mathcal{E}_{\coprod (D_\alpha, c_\alpha, \nu_\alpha)}$$

$\mathcal{LP}$: the Picard groupoid of de Rham factorization lines.

$\mathcal{E}$ - a factorization line

$$X \smallsetminus T \longrightarrow \mathbb{D}$$
$$\cup\!\!\!|$$
$$x \longmapsto \left(D = x;\ c = \emptyset;\ \nu = \frac{dt_x}{t_x}\right)$$

T empty: $\underbrace{\text{this}}_{\text{(the pullback)}}$ is an equivalence of categories (?)

If not: $b \in |T|$ fix a trivialization of $\omega|_{T_b}$

ν_b - triv of $\omega|_{T_b}$ $\quad (0, b, \nu_b) \in \mathbb{D}$

thus $\mathcal{L}_{dR}(X, T) \underset{\sim}{\rightarrow} \mathcal{L}_{dR}(X \smallsetminus T) \times \mathcal{L}^{|T|}$

equivalence of Picard groupoid

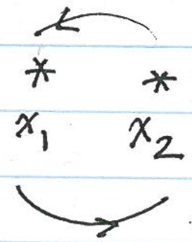
Proof not difficult.

Actually have a line on

$$(X - T) \times \mathbb{G}_m$$

$$(x, \quad c) \longmapsto c\,\frac{dt_x}{t_x} \cdots$$

$$\equiv$$



$$\nu = (t - x_1)^{-1}(t - x_2)^{-1}\,dt$$

Rotate the picture, so that x_1 will get interchanged with x_2.

Monodromy taking into account commutativity constraint for graded lines, becomes ~~?~~ ± 1,

$$\equiv$$

X curve $\quad M$ holonomic $\mathcal{D}$-module $\longmapsto \dfrac{\mathcal{E}_{dR}}{T = \text{sing pt}(M) \text{ with mult.}}$

mult. = order of pole of the connection

$$\mathcal{E}_{dR}(M) \in \mathcal{L}^{\phi}_{dR}(X, T)$$

(X proper; using form)

$$\eta : \mathcal{E}_{DR}(M)(X) \xrightarrow{\sim} \det_{dR} R\Gamma(X, M)$$

$$\mathcal{E}_{dR}(\mathcal{M})_{X \setminus T} = \det{}^{\otimes -1} \mathcal{M} \big|_{X \setminus T}$$

<u>Construction</u> (Deligne's last seminar in Bures)

$(D, c, \nu) \rightsquigarrow$ line $\mathcal{E}_{(D,c,\nu)}$?

? | Assume for simplicity : $c = T$.
| ν is extended to a trivialization of $\omega(D)$

$\mathcal{D}$-module: vector bundle w/ connection outside T.

$$L \subset M\big|_{X \setminus T}$$

a locally free $\mathcal{O}$-submodule $L\big|_{X\setminus T} = M\big|_{X \setminus T}$

$$L_\omega \subset M \otimes \omega$$

1) $\nabla L \subset L_\omega$ (choose any pair; multiply L by high power of parameter...)

2) $\nu L \subset L_\omega$ ∞ dim vect spaces

$$\begin{array}{ccc} M & \xrightarrow{\nabla} & M \otimes \omega \\ \uparrow & & \uparrow \\ L & \longrightarrow & L_\omega \end{array} \qquad M/L \xrightarrow{\overline{\nabla}} M_\omega / L_\omega$$

"polar part".

This complex has fin dim coh

$$M/L \xrightarrow{\overline{\nabla}} M\omega/L_\omega$$

$$\det(\mathrm{Cone}(\overline{\nabla}))$$

$$\boxed{\mathcal{E}_{(D,c,\nu)} = \det \mathrm{Cone}(\overline{\nabla}) \otimes \det\left(L_\omega/\nu L\right)}$$

<u>Claim</u> This line does not depend on any choices involved.

(determinant of an element of the Kallem algebra)

Why is this exactly $\det(R\Gamma_{dR})$?

$$\begin{array}{ccc} M & \xrightarrow{\nabla} & M\omega \\ \cup & & \cup \\ L & \xrightarrow{\nabla} & L_\omega \end{array} \qquad R\Gamma_{dR}(M) = R\Gamma(X, M \xrightarrow{\nabla} M\omega)$$

$$\det R\Gamma_{dR}(M) = \det \mathrm{Cone}(\overline{\nabla}) \otimes$$

$$\otimes \underbrace{\det R\Gamma(X, L \xrightarrow{\nabla} L\omega)}$$

$$\det R\Gamma(X,L)^{\otimes +1} \otimes \det R\Gamma(X, L_\omega)^{\otimes -1}$$

$$\parallel$$

$$\det R\Gamma(X, L \xrightarrow{\nu} L\omega)$$

So we have defined

$$\eta : \mathcal{E}_{dR}(M)(x) \longrightarrow \det R\Gamma_{dR}(M)$$

Have defined everything but the connection.
(can do: 1) geom class field thy; 2) explicitly)

Topological construction

X real analytic surface

$\mathcal{F}$ - constructible sheaf, perfect.
X is compact.
$\mathcal{B}$ - Boolean alg of all constructible subsets
Measure $\mu_{\mathcal{F}}$ on it:

$$\int \mu_{\mathcal{F}} = \chi(X, \mathcal{F})$$

$$\mu_{\mathcal{F}}(P) = \chi(R\Gamma_c(P, \mathcal{F}))$$

P - loc closed constructible

Similar for determinant of cohomology.

What is a measure on $\mathcal{B}$ with values in $\mathrm{Pic}^{\mathbb{Z}}$?

$P = \coprod P_\alpha$ finite disj union

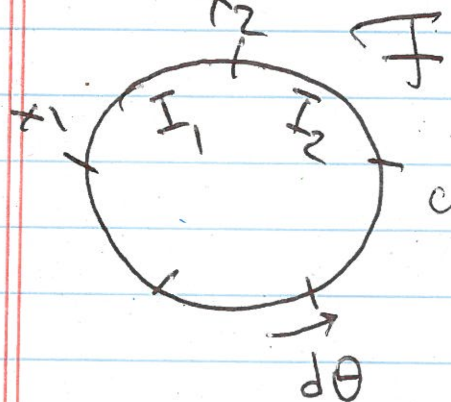
$P \mapsto \mu(P)$ graded superline

$\bigotimes \mu(P_\alpha) \xrightarrow{\sim} \mu(P)$ | subj to...

(extra piece of data)

$$\mu_{\mathcal{F}}(P) := \det R\Gamma_c(P, \mathcal{F}|_P)$$

Beilinson 3



constructible sheaf on S^1

$$\det R\Gamma(\mathcal{F}) = ?$$

finitely many jump points

$$0 \to j_! \mathcal{F}|_{\coprod I_i} \to \mathcal{F} \to \bigoplus \mathcal{F}_{x_i} \to 0$$

$$= \bigotimes_i \det R\Gamma_c(I_i) \otimes \det \mathcal{F}_{x_i} = \qquad \text{(I)}$$

$$= \bigotimes_i \underbrace{\det R\Gamma_c(\leftarrow\!\vert)}_{\mathcal{E}_{x_i}} \qquad \text{(II)}$$

$\mathcal{B}$ – Boolean algebra of constructible sets

$\mu : \mathcal{B} \to \mathcal{L}$ $\qquad \int_X \mu = \det R\Gamma(X, \mathcal{F})$

For $X = S^1$: (I) $\qquad$ What is (II)?

$\underline{\mathcal{B}}_\nu \subset \mathcal{B}(U)$

Def A ν-lens is a loc closed constructible Y s.t.

(i) $\overline{Y}$ is compact in U

(ii) locally in U $\exists N \subset T_U$ s.t. $\nu|_N > 0$

N – cone
(family of $N_x \subset T_x U$, $x \in U$)

$Y = Y_1 - Y_2$ s.t. both Y_1 and Y_2

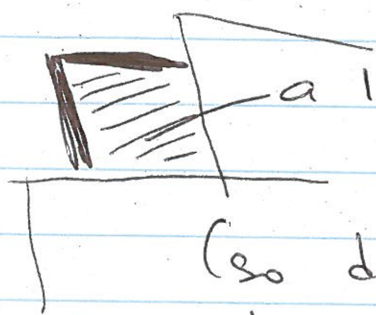
are closed N-invariant

Ex $X = S^1$; $\nu = d\theta$;

(iii) Y shoud be <u>small</u>: on $\overline{Y}$ $\exists f$:
$df = \nu$.

Theorem (Kashiwara-Schapira)

interior $\neq 0$; H° vanishes.



a lens in dim two

(so disjoint unions of lenses ...)

$X \quad \mathcal{F} \qquad T \subset X$. finite subset containing sing points

$U = X \setminus T$

ν — continuous non-vanishing point on U

$$\mathcal{B}_\nu \subset \mathcal{B}(U) \subset \mathcal{B}(X)$$

$\mathcal{B}_\nu^- =$ normalizer of $\mathcal{B}_\nu$ in $\mathcal{B}(X)$

i.e. $\{ V \mid V \cap (\forall \text{set in } \mathcal{B}_\nu) \subset \mathcal{B}_\nu \}$

$$\mathcal{B}_\nu^- / \mathcal{B}_\nu \longrightarrow 2^T$$

$$P \longmapsto P \cap T$$

quotient of boolean algebra by ideal

Claim: ~~this is~~ an isom.

(essentially evident)

The preimage of one point:
draw many lenses around it; throw away their union.

<u>Now</u>: for any sheaf $\mathcal{F}$ s.t. $SS(\mathcal{F}) \cap \nu(X) = \emptyset \dots$

$$\boxed{U \in B_\nu : \ \det R\Gamma(U, \mathcal{F}) \simeq \text{triv}}$$

$\equiv$

X – complex curve; $\mathcal{M}$ – holonomic $\mathcal{D}$-module

Def A theory of ε-factors is a rule that assigns to every (X, M) a de Rham factorization line

<u>Constraints</u>: (a) Local nature (b) Filtration on $\mathcal{M}$: $\mathcal{E}(\mathcal{M}) \xrightarrow{\sim} \mathcal{E}(gr\, \mathcal{M}_i)$
(c) projection formula for finite ~~morj~~ of curves

$$\begin{array}{cc} \mathcal{E} & X' \\ & \big\downarrow{\scriptstyle \pi} \ \text{finite map} \\ \pi_* \mathcal{E} & X \end{array}$$

$$\boxed{\mathcal{E}(\pi_* M) = \pi_*(\mathcal{E}(M))}$$

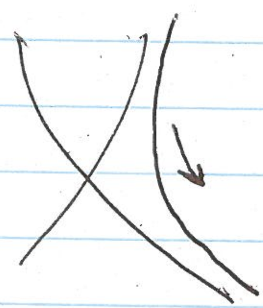
(d) X is compact:

$$\eta : \mathcal{E}(X) \xrightarrow{\sim} \det R\Gamma(X, \mathcal{M})$$

(e) for a family where singularities of $\mathcal{M}$ do not <u>jump</u>:

$\mathcal{E}(\mathcal{M})$ should form a holomorphic family.

(f) One more property: for quadratic degenerations of curves...



$\mathcal{E}$-factors form a <u>gerbe</u>
(factorization lines?)
There is a distinguished automorphisms...

<u>Thm</u> (i) $Aut(\mathcal{E}) = \mathbb{Z}$ $\forall$ theory of $\mathcal{E}$-factors

(ii) all ~~of~~ theories of $\mathcal{E}$-factors are isomorphic

Idea of the proof:

the hard part is to prove the compatibility with $\xrightarrow{\sim} R\Gamma(X, \ldots)$

Discrepancy is a holomorphic function ~~s~~ on the space of...

local systems?... with given local behavior?....

Thm (Some action of the Teichmüller group...) is ergodic.

No nonconstant local systems.

Any holomorphic function on the moduli space depends on the behavior at singularities.

$\frac{dt}{t}$ $L = t\partial_t -$ cov.? lattice

$L_\omega = L \cdot \frac{dt}{t}$

$$\begin{array}{ccc} \mathcal{E}_B & \overset{i_\lambda}{\xrightarrow{\sim}} & \mathcal{E}_{dR} \\ \wr\| & & \wr\| \\ \mathcal{F}_0 & \underset{\sim}{\longrightarrow} & \mathcal{F}_0 \end{array}$$

$i_{\lambda+1} = \lambda\, i_\lambda$

$\lambda =$ eigenval of $t\frac{\partial}{\partial t}$

$i_\lambda \cdot \Gamma(\lambda)$ invariant...

to make it compatible w/ projection formula?...

must satisfy: 1) functional equation for Γ-function...

2) distribution relation for Γ-function...

$\Gamma(\lambda)c_\lambda$ ...

Another way to see Γ-factors: play a little bit with regularized dets.

ε-factors corresp. to irreg sings

||

— " — reg sings

×

(regularized determinants)

(Passage to dif. ops of ∞ orders; dets in Fréchet spaces)

No idea what to do in $\dim X > 1$.