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w <u>Arinkin</u>, <u>M. Habliček</u>

$X, Y \hookrightarrow S$ X, Y, S smooth

$$X \cap Y = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y$$

<u>Serre:</u> $X \cap Y$ has something to do with

$$\mathrm{Tor}_i^{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_Y)$$

$$\mathrm{Tor}_i^S(\mathcal{O}_X, \mathcal{O}_Y) = \mathcal{H}_i(\mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_S} \mathcal{O}_Y)$$

<u>Question:</u> What is $X \overset{R}{\cap}_S X$?

i.e. understand $\mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_S} \mathcal{O}_X$

<u>Thm</u> $\mathcal{H}_i(\mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_S} \mathcal{O}_X) \simeq \Lambda^i N^\vee_{X/S}$

(for local complete intersections)

How far is $\mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_S} \mathcal{O}_X$ from $\bigoplus_{i=0}^{n} \Lambda^i N^\vee_{X/S}[i]$?

If the two were isomorphic, $X \overset{R}{\cap} X = \mathrm{Tot}_X(N[-1])$

Thm (Arinkin, —) Assume char k either 0 or $> \operatorname{codim}_S X$.

TFAE (1) $X \overset{R}{\cap}_S X = \operatorname{Tot}(N[-1])$

(2) N extends to the first infin nbhd of X in S

In fact if E is a v.b. on X, TFAE:

(1) $i^* i_* E \simeq E \otimes S(N^\vee[-1])$

(2) E and N extend to 1st infin nbhd

Mustață: What is the relationship of this with Deligne-Illusie proof of the deg. of Hodge-de Rham?

X smooth, (proj) compact.

Hodge: $H^n_{dR}(X) = \bigoplus_{p+q=n} H^p(X, \Omega^q)$ algebraic

Grothendieck: Algebraic de Rham complex

$$0 \to \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots$$

$$H^n_{dR}(X) \overset{\text{def}}{=} \mathbb{H}^n(X, \Omega^\bullet_{dR,X})$$

Mumford: for a smooth proj in char p, not always $H^n_{dR} \simeq \bigoplus_{p+q=n} H^p(X, \Omega^q_X)$.

Deligne, Illusie: The Hodge-to-deRham

char(k) > ~~p~~ > dim X

$$E_1^{pq} = H^p(X, \Omega^q) \Rightarrow H^{p+q}_{dR}(X)$$

degenerates if X lifts to $\mathbb{Z}/p^2\mathbb{Z} = W_2(\mathbb{F}_p)$.

Main step: $F_*(\Omega^\bullet_{dR,X}) \simeq \bigoplus(\Omega^1_{X'}[1])$

The map d is $\mathcal{O}_{X'}$-linear; $\mathcal{O}_{X'} \subset \mathcal{O}_X$ p^{th} powers.

Mustafé's question: Is there a map $X' \hookrightarrow S$ and a vector bundle E on X' s.t.

$$i^* i_* E \simeq F_*(\Omega^1_{dR/X})^\vee ?$$

Answer. $(X', \mathcal{O}_{X'}) \simeq (X', \mathcal{D}|_{X'}) \hookrightarrow (\mathrm{Tot}\,\Omega^1_{X'}, \mathcal{D})$

~~Eg.~~ In char $p > 0$

$$\mathcal{Z}(\mathcal{D}) = \mathcal{O}_{\mathrm{Tot}\,\Omega^1_{X'}}$$

Ex $X = \mathbb{A}^1$: $\mathcal{Z}(\mathcal{D}) = k[x^p, \partial^p] = \mathcal{O}_{\mathrm{Tot}\,\Omega^1_{\mathbb{A}^1}}$

and $\mathcal{D}$ is Azumaya$/Z(\mathcal{D})$

$\mathcal{D}|_{X'}$ is split as Azumaya alg.

(the diagram above: diagram of twisted spaces...)

Thm (1) $i^* i_* \mathcal{O}_{X'} \simeq (F_* \Omega^1_{dR,X})^\vee$

(2) $\mathcal{O}_{X'}$ lifts to the first infin nbhd

$\Leftrightarrow \mathcal{D}$ splits on the 1^{st} infin nbhd

$\Leftrightarrow X$ lifts to $W_2(k)$.

Ogus-Vologodsky