Cluster algebras can be of finite type $\Longleftrightarrow$ in at least one seed the matrix B "comes from" finite type matrix.
Cartan

Denominators recover positive roots...
Could easily be <u>not</u> finitely generated.

$$\begin{tikzcd} & x_3 \arrow[dr, Rightarrow] & \\ x_1 \arrow[ur, Rightarrow] & & x_2 \arrow[ll, Rightarrow] \end{tikzcd}$$

– <u>not</u> finitely generated (Markov quiver)

(generates solutions to $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 = 0$)

– finite mutation type: 11 exceptional types; a bulk comes from triangulations of curfaces. Bracket: Weyl-Peterson.

$G = GL(n)$

Standard Poisson-Lie structure:

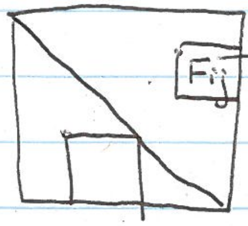
$$\{f_1, f_2\}(X) = R(X\nabla f_1(x)), X\nabla f_2(x)) - $$
$$- R(\nabla f_1(x)\cdot X, \nabla f_2(x)\cdot X)$$

$R = \pi_{\triangledown} - \pi_{\triangle}$ Projections onto upper/lower triang parts.

$$\{x_{ij}, x_{kl}\} = (\mathrm{sign}(k-i) + \mathrm{sign}(l-j))\, x_{il}\, x_{kj}$$

Nowhere near an admissible formula.

Initial cluster:



determinant

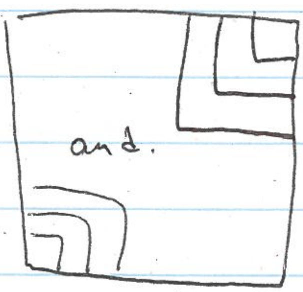
(1) $$\{F_{ij}, F_{k\ell}\} = \overset{\Omega_{ij,k\ell}}{*} F_{ij} F_{k\ell}$$

(2) Recover rules of cluster transformations.

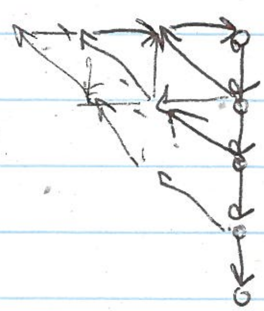
$$\Omega B = [\mathbb{1} \ \ 0]$$

Stable variables:

$2n-1$ of them.



quiver:



(3) transformations of initial cluster variables are regular functions. Based on Plücker relations.

Double Bruhat cells (after F-Z)

$G^{u,v}$ $u, v \in$ Weyl grp $(B_+ u B_+) \cap (B_- v B_-)$

$$\| \\ G^{u,v}$$

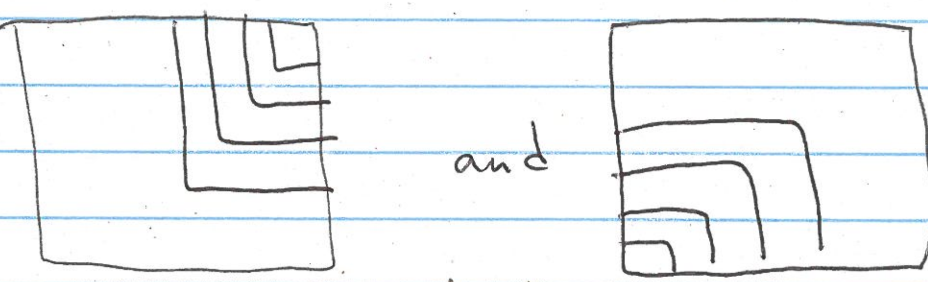
<u>not</u> a cell at all.

w_0 - max length element in W.

G^{w_0, w_0} Zariski open, dense in G.

In $GL(n)$:

G^{w_0, w_0} is characterized by:



they are (collection of ?) open symplectic leaves...

Factorization parameters.

$$w_0 = s_{i_0} \cdots s_{i_\ell} = s_{j_1} \cdots s_{j_\ell}$$

① ② $\cdots$ ⓡ $r =$ rank of G

Consider any shuffle of $(i_1, \ldots, i_\ell)$
$(-j_1, \ldots, -j_\ell)$

Write a product of elements

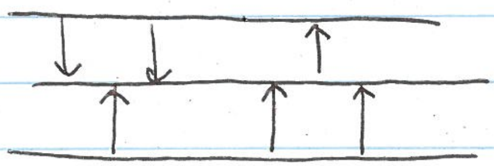
$$i_s \mapsto \exp(t e_{\alpha_s}) \qquad -j_s \mapsto \exp(t e_{-\alpha_s})$$

$$p \mapsto \exp(t\,[f_{\alpha_p}, e_{\alpha_p}])$$
$$\in \mathfrak{h}$$

Claim: the map obtained this way

$$\mathbb{C}^{2\ell+r} \mapsto \text{dense subset of } G^{w_0, w_0}$$

Recall:



two reduced decompositions

Initial cluster: $\underbrace{\text{minors}}_{\text{(in case of } GL(n))}$ that are monomials in parameters t_a.

Generalized minors (Fomin-Zelevinsky)

Principal minors: $X = X_- X_0 X_+$
Gauss decomposition

For any fundamental weight ω

$$X_0^{\omega} = e^{\langle \log X_0, \omega \rangle}$$

Generalized minor "of size k" associated to $u, v \in W$:

$$(u^{-1} X v^{-1})_0^{\omega_k}$$ — fundamental weight

Define functions $\varphi^{\omega}_{u,v} = (u^{-1} X v^{-1})^{\omega}$

$$\{\varphi^{\omega}_{u,vv'}, \varphi^{\omega}_{uu',v}\} = \frac{(Ad_{u'}\omega, \omega') - (\omega, Ad_{v}\omega')}{2} \cdot \text{product}$$

$$\ell(vv') = \ell(v) + \ell(v')$$

$$\ell(uu') = \ell(u) + \ell(u')$$

General rule: by factorization procedure of FZ, produce enough variable as above...

<u>May seem</u>: we have THE cluster structure associated to a simple Lie group.

May not be so.

<u>Want</u>: find a cluster structure compatible with <u>any</u> Poisson-Lie bracket.

Not possible in general. <u>Example</u>:

Simplect Poisson-Lie structure on SL_2 associated to R-matrix

$$r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(triangular) $\qquad$ γ - nilpotent isometry

$\Gamma_{1,2} \supset \Pi$ = simple roots $\qquad \gamma : \Gamma_1 \to \Gamma_2$

<u>Conjecture</u> $\forall$ B.-D. data $\exists$ a compatible cluster structure on G

For different B.-D. data they are <u>not</u> isom.

<u>Evidence</u> (1) Standard P.-L.

(2) SL_3, SL_4

(3) Cremer-Gervais P.-L. structure (farthest removed from the standard one)

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Reason for using R-matrices: completely integrable systems...