SRA and deformations I. Losev

Plan

0) Setting
1) SRA
2) Spherical subalgs
3) Wreath products

0) V fin. dim vect. space$/\mathbb{C}$, $\omega \in \Lambda^2 V^*$ sympl.

fin ~~dim~~ sbgrp $\Gamma \subset Sp(V)$

$\Gamma \curvearrowright S(V)$ symm algebra

$S(V)^\Gamma = \mathbb{C}[V/\Gamma]$ graded, Poisson

$\deg \{,\} = -2$

Goal graded deformations $S(V)^\Gamma$ $\tilde{A}/S(\mathfrak{p})$
assoc alg $\tilde{A}$, graded

$\mathfrak{p}$ - fin dim vector space of parameters; $\deg \mathfrak{p} = 2$

- $\tilde{A}$ is flat/free over $S(\mathfrak{p})$
$\tilde{A}/(\mathfrak{p}) = S(V)^\Gamma$

Today: $\tilde{A}$ via SRA
Next: via Hamiltonian reduction for special groups

<u>Problem</u>: the variety is not smooth

"Remedy": take "resolution". Deform it. Produce deformation of $S(V)^\Gamma$.

Today: "resolution" $S(V)\#\Gamma \underset{\text{as vect. sp.}}{=} S(V)\otimes \mathbb{C}\Gamma$

$$(a_1\otimes g_1)(a_2\otimes g_2) = (a_1\cdot \underbrace{g_1(a_2)}_{\text{image under action.}})\otimes g_1 g_2$$

$S(V)\#\Gamma\text{-mod} = Coh^\Gamma(V) \Rightarrow gl\dim(S(V)\#\Gamma) < \infty.$

SRA = graded deformation of $S(V)\#\Gamma$.

Q. $\exists$? universal SRA $H_{un}/S(\check{\mathfrak{p}}_{un})$ $\forall$ SRA $H/S(\check{\mathfrak{p}})$ $\exists!$ $\check{\mathfrak{p}}_{un}\to\check{\mathfrak{p}}$ s.t.

$$H = S(\check{\mathfrak{p}}) \otimes_{S(\check{\mathfrak{p}}_{un})} H_{un}.$$

§ $\mathfrak{p}_{un}$: <u>Def</u> $\gamma\in\Gamma$ is symplectic reflection if $rk(\gamma - id) = 2$.

$$S = \{\text{Sympl. ref-ns in } \Gamma\} =$$

$$= S_1 \sqcup \ldots \sqcup S_r \text{ - conjugacy classes.}$$

$\mathfrak{p}_{un} := \mathbb{C}^{r+1}$ (assumption: no proper Γ-stable symplectic subspace in V).

Constr. of H_{un}: $s \in S \rightsquigarrow \omega_s \in \Lambda^2 V^*$

$$V = \ker(s - id) \oplus \mathrm{im}(s - id)$$

$$\omega_s = 0 \oplus \omega_{\mathrm{im}(s-id)}$$

Thm (Etingof-Ginzburg '2000)

$$H_{un} = \frac{T(V) \# \Gamma \otimes S(\mathfrak{p}_{un})}{[u,v] = h\omega(u,v) + \sum_{i=1}^{r} c_i \cdot \sum_{s \in S_i} \omega_s(u,v)s}$$

$$u, v \in V$$

where $h, c_1, \ldots, c_r$ is the basis in $\mathfrak{p}_{un}$.

Sketch of proof: Deformations are controlled by $\underbrace{HH^2(S(V)\#\Gamma)_{-2}}_{\text{degree}} = \mathfrak{p}_{un}$

Obstructions: $HH^3(S(V)\#\Gamma)_{\leq -4} = 0$ comp.

$$\Downarrow$$

$$\exists\, H_{un}.$$

$\pi: H_{un} \to S(V)\#\Gamma$

$$\underbrace{/\,\mathfrak{p}_{un}.}_{\deg 2} \Longrightarrow \text{ is iso in deg } 0,\ 1.$$

deg 0: $H_{un,0} = \mathbb{C}\Gamma$

deg 1: $H_{un,1} = V \otimes \mathbb{C}\Gamma \rightsquigarrow V \subset H_{un}$

$$T(V)\#\Gamma \otimes S(\mathfrak{p}_{un}) \to H_{un}$$

actually surjective.

What is the kernel?

$$\text{kernel} = (\mathfrak{p}_{un})_2 = \mathfrak{p}_{un} \otimes \mathbb{C}\Gamma$$

$$[u,v] \rightsquigarrow \exists!\, \kappa: \Lambda^2 V \to \mathfrak{p}_{un} \otimes \mathbb{C}\Gamma$$

$$[u,v] - \kappa(u,v) = 0 \text{ in } H_{un}$$

$\Gamma \subset H_{un} \Rightarrow$ the map above Γ-equivar

$\ker(\pi)$ is generated by $[u,v] - \kappa(u,v)$

because my deformation is <u>flat</u>.

But <u>why</u> is my κ as above? Look at Jacobi identity for u, v, w; $\kappa(u,v)$ <u>cannot</u> include nonreflections.

<u>Result</u>: deformation of $S(V)\#\Gamma$. How to deform $S(V)^\Gamma$?

2). Spherical subalgebra. $e = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma$

$$S(V)^\Gamma = e\big(S(V)\#\Gamma\big)e$$

$$eH_{un}e \subset H_{un} \quad \text{— deformation of } S(V)^\Gamma.$$

<u>Remarks</u> $\Gamma_{refl} \subset \Gamma$ – subgrp generated by symplectic reflections.

$$H_{un}(\Gamma) = H_{un}(\Gamma_{refl}) \#_{\Gamma_{refl}} \Gamma$$

• <u>Why do we care?</u>

0) Deform everything!

1) Related to integrable systems (Calogero-Moser)

2) Interesting representation theory

3) Example of a symplectic reflection group:

$n > 0$: $\Gamma_1 \subset SL_2(\mathbb{C})$ finite subgroup

$$\Gamma_n = S_n \ltimes \Gamma_1^n \circlearrowright (\mathbb{C}^2)^{\oplus n}$$

symplectic vector space w/ form $(\omega_1)^{\oplus n}$

$s_{ij} \in S_n$ transposition

conjugacy class $S_1 =$

$$= \{ s_{ij} \gamma_i \gamma_j^{-1} \mid \gamma \in \Gamma_1, i \neq j \}$$

$\gamma_i :=$ γ in i^{th} copy of Γ_1

<u>More</u>: $\Gamma_1 \setminus \{1\} = \coprod_{i=2}^{r} S_i^0$

decomp. into conjugacy classes.

$$S_i = \{ \gamma_j \mid \gamma \in S_i^0 \}, \; j = 2, \dots, r.$$

<u>McKay corresp.</u> $\Gamma_1 \underset{\text{finite}}{\subset} SL(2, \mathbb{C})$

$N_1, \dots, N_r$ all Γ_1 irreps

Graph. Vertices: $\{1, \dots, r\}$

$$\#(i \longleftrightarrow j) = \dim \mathrm{Hom}(\mathbb{C}^2 \otimes N_i, N_j)$$

Fact: resulting graph = affine Dynkin graph.

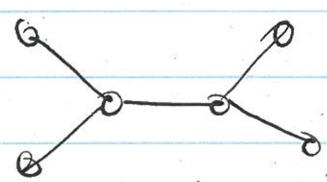
Ex $\Gamma_1 = \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon^{-1} & 0 \end{pmatrix}, \ \varepsilon^6 = 1 \right\}$

dihedral group.

6 irreducibles: $N_1, N_2, N_3, N_4, N_5, N_6$.

dims: 1, 1, 2, 2, 1, 1

McKay graph:



$\widetilde{D}_5$

General result:

$$\text{finite subgroups of } SL_2(\mathbb{C}) / \text{conjugacy}$$
$$\updownarrow \wr$$
$$\text{Affine Dynkin graphs}$$

- 1 is extending vertex (the triv rep)

- $(\dim N_1, \ldots, \dim N_r) = \quad \delta =$ indec imaginary root