Sabbah 1

Fourier Transform

Exponential Sums

$f \in \mathbb{Z}[x_1, \dots, x_n]$

p reduce mod p $\quad f : \mathbb{A}^n_{\mathbb{F}_p} \to \mathbb{A}^1_{\mathbb{F}_p}$

$$\mathrm{Sol}(f, p, t) = \# f^{-1}(t)$$

$\psi_p : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}^\times \qquad \psi(t) = \psi_p(\tau t$

$$\widehat{\mathrm{Sol}}(f, p, \bullet)(\psi) = \sum_{t \in \mathbb{A}^1} \psi(t)\, \mathrm{Sol}(f, p, t) =$$

$$= \sum_{x \in \mathbb{A}^n_{\mathbb{F}_p}} \psi(f(x))$$

$$\boxed{\tau \longmapsto \sum \exp(2\pi i \tau f(x))}$$

Complex analog $\quad f : X \to \mathbb{P}^1$

X smooth complex alg variety

Assume $\omega \in \Gamma(X, \Omega_X^n)$

Start from a CY mfld Y;

$$\exists \quad \begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \\ \mathbb{P}^1 & & \end{array}$$

(i.e. $X \to Y$ - some blow-up)

Integrate along f the (n,n) form $\omega \wedge \bar{\omega}$ as a $(1,1)$ current on $\mathbb{P}^1$:

$$f_*(\omega \wedge \bar{\omega}) : \forall \varphi \in C^\infty(\mathbb{P}^1) \mapsto \int_X (\varphi \circ f)\, \omega \wedge \bar{\omega}$$

This is understood by its Fourier transform (it is itself an analog of the counting fn)

Fix an affine chart: $\underset{t}{\mathbb{A}^1} \hookrightarrow \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$

$$F_t(f_*(\omega \wedge \bar{\omega})) = ?$$

Kernel of Fourier: $\exp(t\tau - \bar{t}\bar{\tau}) =$

$$= \exp 2 \cdot \mathrm{Im}(t\tau)$$

$$F_t\left(f_*(\omega\wedge\bar\omega)\right) = \int_{X_0} e^{\tau f - \overline{\tau f}}\,\omega\wedge\bar\omega$$

$$X_0 = f^{-1}(\mathbb{A}^1)$$ continuous fn in τ

$$\frac{\partial}{\partial\tau}\left(F_t\left(f_*\,\omega\wedge\bar\omega\right)\right) = F_t\left(t\cdot f_*(\omega\wedge\bar\omega)\right)$$

$$\equiv$$

$\overline{\mathbb{Q}}_\ell$-sheaves : ℓ prime $\neq p$

$$f : \mathbb{A}^n_{\mathbb{F}_p} \longrightarrow \mathbb{A}^1_{\mathbb{F}_p}$$

$f_!$ defined for such function

$$f_!\,\mathbb{1}_{\mathbb{A}^n_{\mathbb{F}_p}} = \mathrm{Sol}(f, p, \bullet)$$

Take Trace fns for any constructible $\overline{\mathbb{Q}}_\ell$-sheaves.

K constructible $\overline{\mathbb{Q}}_\ell$-sheaf

$x \in \mathbb{A}^n_{\mathbb{F}_p}$; $\bar{x}$ geom point;

action of $\mathrm{Gal}(K(\bar{x})/K(x))$

$$\boxed{f_!(\mathrm{tr}\, K) = \mathrm{tr}\,(Rf_!\, K)}$$

Complex analogue of a $\overline{\mathbb{Q}}_\ell$-sheaf: holonomic $\mathcal{D}$-module.

$f_*(\omega \wedge \bar{\omega})$ – distribution (or $(1,1)$-current) on $\mathbb{A}^1_{\mathbb{C}}$

$\mathbb{C}[t]\langle \partial_t \rangle$ Weyl algebra of dif ops on $\mathbb{C}[t]$

<u>Prop</u> $f_*(\omega \wedge \bar{\omega})$ is a solution of a nontrivial differential equation. i.e. $\exists\, P \neq 0$ in $\mathbb{C}[t]\langle \partial_t \rangle$

$$P(f_*(\omega \wedge \bar{\omega})) = 0$$

(From that, we will get for free a diff equ for Fourier transform)

Gauss-Manin system of $f: X_0 \to \mathbb{A}^1_{\mathbb{C}}$

Work in the Zariski topology.

Twisted De Rham complex

$$\left(\Omega^{\bullet}_{X_0}[\tau], \, d - \tau\, df \wedge \_\right)$$ is a complex

Rmk

$$\mathbb{C}[t]\langle \partial_t \rangle \qquad [\partial_t, t] = 1$$

acts on each term of the complex, Commutes w/ $d - \tau df$

$$H^k(X_0, \Omega_{X_0}[\tau], d - \tau df)$$

is a <u>module</u> / Weyl algebra

$$\partial_t \left(\sum \eta_j \tau^j\right) = \sum \eta_j \tau^{j+1}$$

$$t \cdot \left(\sum \eta_j \tau^j\right) = \sum_j \left(f\eta_j - (j+1)\eta_{j+1}\right)\tau^j$$

<u>Bernstein's Thm</u> Each $GM^k(f)$ holonomic over $\mathbb{C}[t]\langle \partial_t \rangle$

i.e. <u>any</u> $m \in GM^k(f)$ is annihilated by some nonzero $P \in \mathbb{C}[t]\langle \partial_t \rangle$

P of minimal order: <u>Picard-Fuchs equation</u>

Let $\omega \in \Omega^n(X_0)$

Compute $H^\bullet(X_0, \Omega_{X_0}[\tau], d - \tau df)$ with Čech covering by affine charts

$[\omega] \in GM^n(f)$

$\exists$ Picard-Fuchs equation for ω

$$P(t, \partial_t)\,\omega = 0$$

Let us prove:

$$P \cdot f_*(\omega \wedge \bar{\omega}) = 0$$

Use C^∞ computation for $\mathbb{H}^\bullet(\_\_) = GM^n(f)$

$$H^n\left(\mathcal{A}^\bullet(X_0)[\tau],\ d-\tau df\right)$$

$$\mathcal{P}.\omega = 0 \quad \text{in } GM^n(f)$$

$$\mathcal{P} = \sum \partial_t^k\, a_k(t)$$

$\mathcal{P}_\omega = 0 \iff \exists$ forms $\eta_0, \ldots, \eta_d, \ldots, \eta_{d+m}$

$$\mathcal{P}.\omega = (d-\tau df)\, \mathcal{A}^{n-1}(X_0)[\tau]$$

$$a_0 \circ f\ \omega = d\eta_0$$

$$\vdots$$

$$a_d \circ f \cdot \omega = d\eta_d - df \wedge \eta_{d-1}$$

$$0 = d\eta_{d+1} - df_n \cdot \eta_d$$

$$\vdots$$

$$0 = \ldots - df \wedge \eta_{d+m}$$

To conclude that $\mathcal{P}.\, f_*(\omega \wedge \bar{\omega}) = 0$ one uses Stokes formula:

$$\int d\left[(\varphi\circ f)\,\eta\wedge\psi\right]=0$$

η is a $(n-1,0)$-form

ψ is a $(0,n)$-form

$\varphi\in C_c^\infty(\mathbb{A}^1)$ test function

The rest of the computation is in the notes.

→ Consider the holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-mod

$$\mathbb{C}[t]\langle\partial_t\rangle/(P)$$

Fourier transform of a $\overline{\mathbb{Q}}_\ell$-sheaf/ $\mathbb{C}[t]\langle\partial_t\rangle$-mod

Corr. between Deligne, Malgrange, Ramis.
Intro by Deligne

In late '70s: considered irreg singularities as a pathology
Changed his mind: found analogy between

$\mathcal{O}$, $d + dx$ on $\mathbb{A}^1$ $\xleftrightarrow{?}$ ℓ-adic sheaves $\mathcal{L}(\psi)$ from Artin-Schreier cov. of $\mathbb{A}^1_{\mathbb{F}_p}$

(horiz. section e^{-x})

Artin-Schreier covering $\mathbb{A}^1_x \to \mathbb{A}^1_t$

$$x^p - x = t$$

define for each additive character of $\mathbb{Z}/p\mathbb{Z}$:

$\mathcal{L}(\psi)$ lisse rk 1 ℓ-adic sheaf on $\mathbb{A}^1_{\mathbb{F}_p}$

$$\begin{tikzcd}
 & \mathbb{A}^1_{t\,\mathbb{F}_p} \times \hat{\mathbb{A}}^1_{\tau\,\mathbb{F}_p} \arrow[dl, "\pi"'] \arrow[dr, "\hat{\pi}"] \arrow[r, "\mu"] & \mathbb{A}^1_{\mathbb{F}_p} \\
\mathbb{A}^{\wedge}_{\mathbb{F}_p} & & \hat{\mathbb{A}}^1_{\mathbb{F}_p}
\end{tikzcd}$$

$(t,\tau) \mapsto t\tau$

$\mathcal{F}$ $\overline{\mathbb{Q}}_\ell$-sheaf

$$\begin{tikzcd} \mathbb{A}^1_{\mathbb{C}} & \mathbb{A}^1_{\mathbb{C}} \times \hat{\mathbb{A}}^1_{\mathbb{C}} \arrow[l, "\mu"'] \arrow[dl, "\pi"'] \arrow[dr, "\hat{u}"] & \\ \mathbb{A}^1_{\mathbb{C}} & & \hat{\mathbb{A}}^1_{\mathbb{C}} \end{tikzcd}$$

$\mathcal{M}$ holonomic

$\mathbb{C}[t]\langle\partial_t\rangle$ - mod

$$\mathcal{F}_{!,\psi}(\mathcal{F}) = R\pi_!\left(L\pi^* F \otimes \mu^* \mathcal{L}(\psi)\right)$$

$$\mathcal{F}_{*,\psi}(\cancel{\mathcal{A}}) = R\pi_* \ \ldots$$

Laumon: they are <u>equal</u>.

$$\mathcal{F}(\mathcal{M}) = R\pi_+\left(\underbrace{\overline{\pi^+ \mathcal{M}} \otimes \mathcal{E}^{t\tau}}_{\pi^*\mathcal{M} \text{ as } \mathbb{C}[t,\tau]\text{-mod}}\right)$$

<u>Connection</u>: $\pi^*\nabla + d(t\tau)$

Push for

Sabbah 2

$$\begin{tikzcd} & \mathbb{A}^1_{\mathbb{C}} \times \hat{\mathbb{A}}^1_{\mathbb{C}} \arrow[dl, "\pi"'] \arrow[dr, "\hat{\pi}"] & \\ \mathbb{A}^1_{\mathbb{C}} & & \hat{\mathbb{A}}^1_{\mathbb{C}} \end{tikzcd}$$

$$\mathcal{F}M = \hat{\pi}_+ \left(\pi^+ M \otimes E^{t\tau} \right)$$

$\pi^+ M = \mathbb{C}[\tau] \otimes_{\mathbb{C}} M \qquad 1 \otimes \nabla \pm d \otimes 1$

$\left(\pi^+ M \otimes E^{t\tau} \right)$: same object
twisted connection

$$1 \otimes \nabla + d \otimes 1 + d(t\tau) = \tilde{\nabla}$$

$\hat{\pi}_+$:

$$\mathbb{C}[\tau] \otimes M \xrightarrow{\tilde{\nabla}_{\partial/\partial t}} \mathbb{C}[\tau] \otimes M$$

the cohomology of ~~this~~ complex is ... $\hat{\pi}_+ \ldots$

$$\mathbb{C}[\tau] \otimes M \xrightarrow{1 \otimes \nabla + \tau \otimes 1} \mathbb{C}[\tau] \otimes M$$

injective

$\mathrm{Coker} \xrightarrow[\sim]{\tau^k \otimes m \mapsto (-\nabla_{\partial_t})^k \cdot m} M$ as $\mathbb{C}$-vector spaces;

$\mathbb{C}[\tau]\langle \partial_\tau \rangle$ on the coker :
τ acts as $-\partial_t$; ∂_τ $-t$

So two ways of looking at Fourier:

- Via integral transform (passing to dim 2)
- Via an action of $\mathbb{C}[t]\langle\partial_t\rangle$ twisted by automorphism.

<u>Stationary phase formula</u> $f: X \to \mathbb{P}^1$

$$F_t(f_*(\omega\wedge\bar{\omega})) = \int_{X_0} e^{\tau f - \bar{\tau}\bar{f}} \omega\wedge\bar{\omega}$$

moderate growth (because temperate).
Controls $|F_t(f_*(\omega\wedge\bar{\omega}))|$

the function is a superposition:

$$e^{2i(\mathrm{Im}(c\cdot\tau))} \times fn \text{ with nonose. phase}$$

These $c \in \mathbb{C}$ are critical values of f

We will deduce that from the Fourier transform of the Picard-Fuchs operator.

$P \in \mathbb{C}[t]\langle\partial_t\rangle$ Picard-Fuchs op of ω

$M = \mathbb{C}[t]\langle\partial_t\rangle/(P)$

$F_M = \mathbb{C}[\tau]\langle\partial_\tau\rangle/({}^F P)$

<u>Pb</u> Understand the formal behaviour of ${}^F M$ near $\tau = \infty$.

Easier to understand in the *formal* parameter $\tau' = 1/\tau$

M on $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$

$$\mathcal{M} = j_+ M \text{ as } \mathcal{D}_{\mathbb{P}^1}\text{-module}$$

$$\parallel$$

$$j_* M \text{ with meromorphic connection.}$$

By def $\mathcal{O}_{\mathbb{P}^1}(*\infty) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{M} = \mathcal{M}$

$$\hat{\mathcal{M}} = \hat{\mathcal{O}}_{\mathbb{P}^1, c} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{M}$$

is a $\mathbb{C}[\![t_c]\!]\langle \partial/\partial t_c \rangle$ - module

$t_c = t - c,\ c \in \mathbb{A}^1 \qquad t_c = 1/t,\ c = \infty$

$\partial_{t_c} = \partial_t,\ c \in \mathbb{A}^1 \qquad -t^2 \partial_t,\ c = \infty$

$^F\mathcal{M}$ on $\hat{\mathbb{P}}^1$, $^F\hat{\mathcal{M}}$?

Known: $\underbrace{GM^n(f)}_{M}$ $\begin{cases} \underline{\text{regular sing}} \text{ on } \mathbb{A}^1 \cup \{\infty\} \\ \underline{\text{holonomic}} \end{cases}$

$\Downarrow$

^FM has only sing at $\tau = 0$ $\underline{\text{reg}}$, $\tau = \infty$ $\underline{\text{irreg}}$

$^F\mathcal{M}$: so far define $^F M$
and then $^F\mathcal{M} = j_*({}^F M)$

OR: interprete $d(t\tau)$ as merom form on $\mathbb{P}^1$.

Now $^F\mathcal{M} = \hat{\pi}_+(\pi^+\mathcal{M} \otimes \mathcal{E}^{t\tau})$

Ex $\int_{X_0} e^{\tau f - \overline{\tau f}} \omega \wedge \bar{\omega}$

$$g : X \to \mathbb{P}^1$$

Consider $\int_{X_0} e^{\tau f + g - \overline{\tau f + g}} \omega \wedge \bar{\omega}$

Local Fourier transform

Understand the contribution of a singular point of M to $^F\hat{\mathcal{M}}_\infty$.

Analogue for l-adic sheaves - Laumon '87
→ product formula

Complex setting (Beilinson - Bloch - Esnault - Deligne)
not published.

Questions: Given M

1) to compute the singular pts of ${}^F M$
2) at each singular point c of ${}^F M$ to understand ${}^F\hat{\mathcal{M}}^{\mathcal{E}}_c$ from data of M

Local Fourier transform

- ℓ-adic setting (Laumon): defined in terms of "vanishing cycles"

- Complex setting: microlocalization (formal)

$$\mathcal{E}^{(c,\infty)} \longrightarrow \text{give contribution of } c \in \mathrm{Sing}(M) \text{ to } {}^F\hat{\mathcal{M}}_\infty .$$

t_c = local coordinate at c

say, $t \in \mathbb{A}^1$; $t_c = t - c$

$$(c,\infty) = \Big\{ \sum_{i \le i_0} a_i(t_c)\cdot \eta^i \ \Big/\ a_i(t_c) \in \mathbb{C}[\![t_c]\!] \Big\}$$

$$P \circ Q = \sum_{\alpha \ge 0} \frac{1}{\alpha!}\, \partial^\alpha_\eta P \cdot \partial^\alpha_{t_c} Q$$

$$\mathbb{C}[t]\langle \partial_t \rangle \longrightarrow \mathcal{E}^{(c,\infty)}$$

$$t \longmapsto t_c + c \qquad \partial_t \to \eta$$

$$\mathcal{E}^{(\infty,\infty)} : \left\{ \sum a_i(t_\infty)\eta^i \right\} \quad a_i \in \mathbb{C}[\![t_\infty]\!]$$

η ...

$$\mathbb{C}[t_\infty]\langle \partial_{t_\infty} \rangle \longmapsto \mathcal{E}^{(\infty,\infty)}$$

$$t_\infty \longmapsto t_\infty$$

$$\partial_{t_\infty} \longmapsto \eta$$

<u>Thm</u> (Garcia Lopez)

$$^{F}\widehat{\mathcal{M}}_\infty = \underbrace{\mathbb{C}((\tau_\infty))}_{\substack{\| \\ 1/\tau \\ \|}} \otimes_{\mathbb{C}[\tau]} F_M$$

$$\underbrace{\bigoplus_{c \in \mathbb{P}^1} \mathcal{E}^{(c,\infty)} \otimes_{\mathbb{C}[\![t_c]\!]\langle \partial_{t_c} \rangle} \widehat{\mathcal{M}}_c}_{\left(\substack{\text{local fourier transforms} \\ \mathcal{F}^{(c,\infty)} \widehat{\mathcal{M}}_c}\right)} = \bigoplus_{c \in \mathbb{P}^1} \left(\mathcal{E}^{(c,\infty)} \otimes_{\mathbb{C}[t]\langle \partial_t \rangle} M \right)$$

How to understand (give a formula)?

<u>Rmk</u> If c is not a sing pt of M and not ∞, then $\mathcal{E}^{(c,\infty)} \otimes \widehat{\mathcal{M}}_c = 0$

as modules/ $\mathbb{C}((\tau_\infty))\langle\partial_{\tau_\infty}\rangle$

Each $\hat{\mathcal{M}}_c$ can be defined explicitly

(Turritin-Levelt thm)

$\hat{\mathcal{M}}_c^{(\lambda)}$ $\lambda \in \mathbb{Q}_+$ slope

$$\hat{\mathcal{M}}_c = \bigoplus_\lambda \hat{\mathcal{M}}_c(\lambda)$$

$\hat{\mathcal{M}}^{(\lambda)}$ has a decomposition into ~~simple~~ elementary objects

Elem. objects described by data of:

* regular meromorphic connection R, ∇ $\quad R = \mathbb{C}((t_c))^N$

$$\nabla = d + A\frac{dt_c}{t_c}$$

A – constant matrix

* $\varphi \in \mathbb{C}((t_c^{1/p}))/\mathbb{C}[[t_c^{1/p}]]$

polar part of a ramified function.

Given by a simple explicit formula in the notes.