Schedler 1

a) Poisson bracket $\{,\}$ on $\mathcal{O}_X$

<u>Examples</u> T^*Y; $\mathbb{C}^{2n}$; $\operatorname{Sp} G V$
G finite

$$X = V/G$$

$\mathfrak{g}^*$; coadjoint orbits;

X Poisson variety. Z Poisson subvar if I_Z is a Poisson ideal; $\{I_Z, \mathcal{O}_X\} \subset I_Z$ (Poisson scheme...)

Particular case: $\mathfrak{g}$ semisimple; $\operatorname{Nil}(\mathfrak{g}) \subseteq \mathfrak{g}$

Killing: $\mathfrak{g} \simeq \mathfrak{g}^*$ $\quad \operatorname{Nil}(\mathfrak{g}) = \overline{G \cdot e}$

e "principal nilpotent"

$\operatorname{Nil}(\mathfrak{g})$ singular.

Symplectic resolutions: $\widetilde{X} \to X$

proper, Poisson

$\widetilde{X}$ symplectic

<u>Ex</u> $T^*(G/B) \to \operatorname{Nil}(\mathfrak{g})$

$B \subset G$ Borel

$\mathbb{C}^{2n}/T \quad T \subset \operatorname{Sp}(2n)$

Particular case: $\mathbb{C}^2$

cyclic $\left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \right\rangle$ dihedral $\left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$

$$\zeta = e^{2\pi i/n}$$

$$SL(2,\mathbb{C}) \supseteq SU(2,\mathbb{C}) \twoheadrightarrow \underset{\cup}{SO(3,\mathbb{R})}$$

rotation groups of Platonic solids (3 of them: tetrah., cube $\Leftrightarrow$ octah., dodec. $\Leftrightarrow$ icos.)

<u>Poisson traces</u> $\mathcal{O}_X \xrightarrow{\varphi} \mathbb{C}$ X - affine Poisson

$$\varphi(\{f,g\}) = 0$$

$$HP_0(\mathcal{O}_X) = \mathcal{O}_X / \{\mathcal{O}_X, \mathcal{O}_X\} \qquad \{\text{P. traces}\} = HP_0(\mathcal{O}_X)^*$$

affine symplectic case:

$$HP_0(\mathcal{O}_X) \xrightarrow{\sim} H_{DR}^{\dim}(X)$$

$$[f] \longmapsto f \cdot \mathrm{vol}_X \in \Omega^{\dim X}(X)$$

$$HP_0(\mathcal{O}_{\mathfrak{g}^*}) = (\mathrm{Sym}\,\mathfrak{g})_{\mathfrak{g}} = (\mathrm{Sym}\,\mathfrak{g})^G = \mathcal{O}_{\mathfrak{g}^*} /\!/ G$$

So $\overline{G\cdot e} \subseteq \mathfrak{g}^* \; : \; e \in \mathfrak{g}^*$

$$HP_0(\mathcal{O}_{\overline{G\cdot e}}) = \mathcal{O}_{\overline{G\cdot e}/\!/G} \cong \mathbb{C}$$

<u>Comment</u> X Poisson, $\mathbb{C}^*$ contracting action
i.e. $\mathcal{O}_X$ nonneg graded
(as a <u>Poisson</u> alg)

$$\mathcal{O}_X \xrightarrow[\text{proj}]{} (\mathcal{O}_X)_0 \; (= \mathbb{C})$$

Poisson ~~trace~~

If $\{x\} \subset X$ 0-dim sympl leaf,

$ev_x : \mathcal{O}_X \to \mathbb{C}$ is a Poisson trace

<u>Nontrivial</u>: At $e \in \mathfrak{g}$, $\mathfrak{g}$ f.d. s.-s.,

there is a Kostant Slodowy slice
$e \in S_e \subseteq \mathfrak{g}$, transverse to ~~$S_e$~~ $G\cdot e$

$$T^*G/B \overset{\rho}{\twoheadrightarrow} \underset{\substack{\uplus\\ e}}{\mathrm{Nil}(\mathfrak{g})} = \mathcal{N}$$

$$\rho^{-1}(S_e \cap \mathcal{N}) \twoheadrightarrow S_e \cap \mathcal{N}$$

symplectic resolution

fact: this is a symplectic resolution.
For the appropriate Poisson structures that you can get by Hamiltonian reduction.

Thm (ES)

$$HP_0\left(\mathcal{O}_{S_e \cap N}\right) \cong H^{\dim_{\mathbb{C}}(S_e \cap N)}(\rho^{-1}(S_e \cap N)) \cong H^{top}(\underbrace{\rho^{-1}(e)}_{\text{Springer fiber}})$$

Conjecture: This holds for any sympl resolution with X affine:

$$HP_0(\mathcal{O}_X) \simeq H^{top}(\tilde{X})$$

Theorem: If X has finitely many symplectic leaves then $HP_0(\mathcal{O}_X)$ is f.d.

Dfn: X has fin. many sympl. leaves if

$$X = \bigsqcup_{fin} X_i, \quad \underline{\text{locally closed}}, \quad \overline{X_i} \subset X \text{ Poiss}$$

X_i symplectic

Ex $\tilde{X} \to X$ sympl res $\Rightarrow$ fin many s. leaves

$\mathbb{C}^{2n}/G \quad G < Sp(2n)$

Closures of Sympl. leaves are

$$(\mathbb{C}^{2n})^K \Big/ \left(N(K)/K\right)$$

~~K ≤ A~~ $K \subset G$ parabolic

Def $K \subset G$ parabolic if $\exists\, v \in \mathbb{C}^{2n}$ $Stab_G(v) = K$

Generally, $\dim V \geq 2$, V symplectic v.s.

V/G <u>does not</u> admit symplectic resolution except in special cases.

<u>To prove theorem</u>: rewrite $HP_0(\mathcal{O}_X)$ using diff. equs. $f \in \mathcal{O}_X$; $\xi_f = \{f, -\}$

Hamiltonian vector field $\in H(X)$

$$HP_0(\mathcal{O}_X)^* = \cancel{\mathrm{Hom}}\ (\mathcal{O}_X^*)^{H(X)}$$

solution space of d.e.

Dfn X smooth,

$$1 \in \mathcal{M}(X) := \mathcal{D}_X \,/\, H(X) \cdot \mathcal{D}_X$$

$$HP_0(\mathcal{O}_X) = \mathcal{M}(X) \otimes_{\mathcal{D}(X)} \mathcal{O}_X$$

$$\boxed{\begin{aligned} HP_0(\mathcal{O}_X)^* &= \cancel{\mathcal{M}(X)} \\ &= \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}(X), \mathcal{O}_X^*) \end{aligned}}$$

Thm $\pi : V \to pt$ $\quad \pi_0(M) = M \otimes_{\mathcal{D}_V} \mathcal{O}_V$

Thm If X has fin dim sympl leaves then $\mathcal{M}(X)$ is holonomic.