The Heisenberg–Weil Representation, Geometrization, and Applications

Speaker: Shamgar Gurevich, UW Madison.

Lecture I: The Heisenberg–Weil Representation and Fast Wireless Communication

Abstract: The space $\mathcal{H} = \mathbb{C}(\mathbb{Z}_N)$ of complex valued functions on $\mathbb{Z}_N = \{0, ..., N-1\}$ is called in electrical engineering the Hilbert space of digital signals. An important task for doing efficient wireless communication is the algorithmic construction of 'good signals' in $\mathcal{H}$. In lecture I, I will explain how to use the Heisenberg–Weil representation, i.e., the natural symmetries of the space $\mathcal{H}$, to construct 'good signals'. These signals will enable us to present a new algorithm, called the "Flag Method", that suggests a solution to the following problem: You transmit your 'good signal' $S \in \mathcal{H}$, to the antenna; the antenna receives the signal $R \in \mathcal{H}$ which has the form

$$R[n] = e^{\frac{2\pi i \omega_0 n}{N}} \cdot S[n + \tau_0] + W[n], \quad n \in \mathbb{Z}_N,$$

where $W \in \mathcal{H}$ is a random signal, $\tau_0 \in \mathbb{Z}_N$ encodes the time took the signal to arrive to the antenna, and $\omega_0 \in \mathbb{Z}_N$ encodes your radial velocity with respect to the antenna.

**Problem (Channel Estimation)** Design $S \in \mathcal{H}$, and a method to efficiently extract $(\tau_0, \omega_0)$ using $R$ and $S$.

In my lecture I first introduce the classical matched filter algorithm that suggests the 'traditional' way (using fast Fourier transform) to solve the channel estimation problem in order of $O(N^2 \log(N))$ arithmetical operations. Then I will explain how the flag method solves this problem in a much faster way of order of $O(N \log(N))$ operations (in certain applications $N \gg 1,000$).

Finally, I will explain applications of our method to mobile communication of fast moving users, and to global positioning system (GPS).
This is a joint work with A. Fish (Math, Madison), R. Hadani (Math, Austin), A. Sayeed (EE, Madison), and O. Schwartz (EECS, Berkeley).

Lecture II: The Geometric Weil Representation

Abstract: This is a sequel to Lecture I. It will be more specialized. The finite Weil representation is the algebra object that governs the symmetries of the Hilbert space \( \mathcal{H} = \mathbb{C}(\mathbb{Z}_N) \). The main objective of this talk is to introduce the geometric Weil representation which is an algebra-geometric (\( \ell \)-adic perverse Weil sheaf) counterpart of the finite Weil representation. Then, I will explain how the geometric Weil representation is used to prove the main technical results stated in Lecture I. In the course, I will explain the Grothendieck geometrization procedure by which sets are replaced by algebraic varieties and functions by sheaf theoretic objects.

This is a joint work with R. Hadani (Math, Austin).

Lecture III: Fast Fourier Transform

Abstract: The Discrete Fourier Transform (DFT)

\[
\hat{f} = DFT(f), \quad f = \begin{pmatrix}
    f(0) \\
    \vdots \\
    f(N - 1)
\end{pmatrix}, \quad DFT = \frac{1}{\sqrt{N}} \left( e^{\frac{2\pi i}{N} \cdot \tau \cdot \omega} \right)_{0 \leq \tau, \omega \leq N - 1}
\]

is one of the most important operators in computational mathematics. In particular, I made use of it in the flag algorithm presented in Lecture I. The DFT operator acts on the \( N \)-dimensional Hilbert Space \( L^2(\mathbb{Z}_N) \) of complex valued functions on the group of integers modulo \( N \). It becomes very useful in the last century due to the Cooley–Tukey Fast Fourier Transform (FFT) algorithm that computes the DFT in order of \( O(N \cdot \log(N)) \) arithmetic operations. In the lecture I will elaborate on an idea—due to Auslander and Tolimieri—which establishes this algorithm as a logical consequence of the construction of an arithmetic model which realizes the irreducible representations of Heisenberg group.

This is a joint work with R. Hadani (Math, Austin).