The asymptotic behaviour of doubly periodic instantons and Stokes structure

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Introduction

Let $T$ be an elliptic curve over $\mathbb{C}$. For $\lambda \in \mathbb{C}$, let $\mathcal{M}^\lambda$ denote the moduli space of line bundles of degree 0 with a flat $\lambda$-connection.

\[ \mathcal{M}^\lambda := \{ (L, \mathbb{D}^\lambda) \mid L \in \text{Pic}_0(T), \mathbb{D}^\lambda : \lambda\text{-connection of } L \}/ \sim \]

A flat $\lambda$-connection is a differential operator $\mathbb{D}^\lambda : L \to L \otimes \Omega^1_X$ such that

(i) $\mathbb{D}^\lambda(f s) = \mathbb{D}^\lambda f \otimes s + (\lambda \partial s + \mathcal{D}_s) f \otimes s$ for $f \in \mathcal{C}^\infty(T), s \in \mathcal{C}^\infty(T, L)$,
(ii) $\mathbb{D}^\lambda \mathbb{D}^\lambda = 0$.

The space $\mathcal{M}^\lambda$ is an affine space bundle over $T^\nu := \text{Pic}_0(T)$.

Goal

1. The behaviour of holomorphic vector bundles on $\mathcal{M}^\lambda$ around $=0$. (Hukuhara-Turrittin type theorem, Stokes structure,...)
2. Application to instantons on $T^\nu \times \mathbb{C}$.

Vector bundles on $\mathcal{M}^\lambda$

Let $T = \mathbb{C}/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}$. Then $T^\nu := \text{Pic}_0(T) \cong \mathbb{C}/\Lambda^\nu$, where $\Lambda^\nu := \{ z \in \mathbb{C} \mid \text{Im}(z) \in \mathbb{R}, \forall \xi \in \Lambda \}$.

The identification is induced by $\mathbb{C} \ni z \mapsto (\zeta + \lambda \partial \zeta, \zeta) \mapsto \zeta \in \Lambda^\nu$.

$\mathcal{M}^\lambda$ is described as the quotient $\mathcal{M}^\lambda \cong \{ (\xi, \eta) \in \mathbb{C}^2 \}/\sim$,

\[ (\xi, \eta) \sim (\xi + \chi, \eta - \lambda \chi) \quad \forall \chi \in \Lambda^\nu \]

The identification is induced by $\mathbb{C} \ni (\xi, \eta) \mapsto (\zeta + \lambda \partial \zeta, \zeta) \mapsto \zeta \in \Lambda^\nu$.

The isomorphism is induced by $\rho_0(\xi) = \exp(2\sqrt{-1}\text{Im}(\xi^2)) = (\zeta^2 - \mathcal{D}_\zeta \zeta)$ on $T$.

The fibration $\mathcal{M}^\lambda \to T^\nu$ is given by $\zeta \mapsto (\xi, \eta)$.

Hitchin transform

Let $X \subset \mathbb{C}$ be an open subset. We obtain an open subset $\mathcal{V}_0^\lambda(X) \subset \mathcal{M}^\lambda$.

$\mathcal{M}^\lambda$ is an affine space bundle over $T^\nu \times \mathbb{C}$.

We use the natural coordinate $(z, w)$. We have a natural diffeomorphism $\mathcal{M}^0 \simeq \mathcal{M}^\lambda$ given by

\[ (\xi, \eta) = (\xi + \lambda \partial \zeta, \zeta) \mapsto (\xi, \eta) \mapsto (\xi + \lambda \partial \zeta, \zeta) \mapsto \zeta \in \Lambda^\nu \]

The isomorphism is induced by $\rho_0(\xi) = \exp(2\sqrt{-1}\text{Im}(\xi^2)) = (\zeta^2 - \mathcal{D}_\zeta \zeta)$ on $T$.

The fibration $\mathcal{M}^\lambda \to T^\nu$ is given by $\zeta \mapsto (\xi, \eta)$.

3-flat bundle

A $3$-flat bundle on a complex manifold $X$ is a $\mathcal{C}^\infty$-bundle $V \to X$ with a differential operator $D^3 : V \to \Omega^3_X$ such that

\[ D^3(f s) = f D^3 s + (\lambda \partial s + \mathcal{D}_s) f \otimes s \quad f \in \mathcal{C}^\infty(X), s \in \mathcal{C}^\infty(X, V) \]

If $X \subset \mathbb{C}$, a flat $\lambda$-connection is given by commutative actions $D^\lambda_X$ and $D^\lambda_{\mathbb{C}}$ satisfying

\[ D^\lambda_X(f s) = f D^\lambda_X s + \lambda \partial s, \quad D^\lambda_{\mathbb{C}}(f s) = f D^\lambda_{\mathbb{C}} s + \mathcal{D}_s f \otimes s. \]
Holomorphic vector bundle on \( \Psi_0^{-1}(X) \Rightarrow \) flat \( \lambda \)-connection on \( X \)

From a holomorphic vector bundle \( E \), we obtain a \( \mathcal{E}_X \)-module \( \Psi_0(E) \) on \( X \):

\[
\Psi_0(E)(U) = \{ C^\infty\text{-sections of } E \text{ on } \Psi_0^{-1}(U) \} \quad (U \subset X \text{ open})
\]

It is equipped with the actions of \((1 + |\lambda|^2)\partial_\lambda \) and \((1 + |\lambda|^2)\overline{\partial}_\eta \). They give a flat \( \lambda \)-connection of \( \Psi_0(E) \).

It can be regarded as “a \( \lambda \)-flat bundle of infinite rank”.

We have a natural inclusion

\[
(V, D_\lambda) \subset \Psi_0^\ast \Psi_0^\ast (V, D_\lambda) \cong (V, D_\lambda) \otimes \Psi_0^\ast (\mathcal{O}_\lambda).
\]

The analogy of holomorphic vector bundles on \( \mathcal{M}^\lambda \) and \( \lambda \)-flat bundles on \( \mathcal{C} \) can be more acute around \( \infty \).

Recall \( \mathcal{M}^\lambda \rightarrow T^\infty \) is affine space bundle given by \((\xi, \eta) \mapsto \xi \).

We obtain the natural projective completion \( \overline{\mathcal{M}}^\lambda \), by adding \( \eta = \infty \).

Let \( T^\lambda_\infty \) denote \( \{ \eta = \infty \} \), which is naturally isomorphic to \( T^\infty \).

\[
\overline{\mathcal{M}}^\lambda = \mathcal{M}^\lambda \cup T^\lambda_\infty \quad (\text{set theoretically})
\]

- We will consider vector bundles \( E \) on a neighbourhood of \( T^\lambda_\infty \) such that \( E|_{T^\lambda_\infty} \) is semistable of degree 0.
- \( E \) has a kind of Stokes structure, if \( \lambda \neq 0 \).
  (The case \( \lambda = 0 \) is simpler.)

Example

Let \( C \cong \mathbb{C}/\Lambda \). We use the standard coordinate \( z \) of \( C \).

A finite dimensional \( \mathbb{C} \)-vector space \( V \) induces a \( \mathbb{C}^\infty \)-bundle \( \mathcal{V} := V \times \mathbb{C} \) over \( C \). It has a natural holomorphic structure

\[
\overline{\mathcal{V}}_0 : \mathcal{C}^\infty(C, \mathcal{V}) \longrightarrow \mathcal{C}^\infty(C, \mathcal{V} \otimes \Omega^1_{\mathbb{C}})
\]

\[ f \in \text{End}(\mathcal{V}) \] gives a holomorphic structure \( \overline{\mathcal{V}}_0 + f d\mathcal{V} \) of \( \mathcal{V} \).

**Lemma** \((\mathcal{V}, \overline{\mathcal{V}}_0 + f d\mathcal{V})\) is semistable of degree 0.

Conversely, any semistable vector bundle of degree 0 can be expressed as above (not uniquely).

Let \( E_0 \) be a semistable bundle of degree 0 on \( C \). We have the Fourier-Mukai transform \( \text{FM}(E_0) \) on \( \mathcal{C}^\prime = \text{Pic}_0(C) \).

**Fourier-Mukai transform (the simplest case)**

We have the universal line bundle \( \mathcal{L} \) (Poincaré bundle) on \( C \times \mathcal{C}^\prime \).

Let \( \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2 : C \times \mathcal{C}^\prime \rightarrow \mathcal{C}^\prime \) be the projections.

For an \( \mathcal{O}_C \)-module \( M \), we obtain \( \text{FM}(M) := p_2 \ast (p_1 \otimes \mathcal{L}) \ast M \) in \( D^b(\mathcal{C}^\prime) \).

If \( M \) is a semistable bundle of degree 0, \( \text{FM}(M) \) is a torsion \( \mathcal{O}_C \)-module.

Let \( t : \mathcal{C}^\prime \longrightarrow \mathcal{C}^\prime \) be given by \( t(\xi) = -\xi \). We set

\[
s(E_0) := \text{the support of } t^\ast \text{FM}(E_0)
\]

If \( E_0 = (\mathcal{V}, \overline{\mathcal{V}}_0 + f d\mathcal{V}) \), \( s(E_0) = \{ \text{the eigenvalue of } f \text{ modulo } \Lambda^\prime \} \).

\((\mathcal{V}, f)\) is unique up to isomorphisms, once we fix a lift of \( s(E_0) \) to \( \mathcal{C}^\prime \).
An equivalence

Let $\tilde{s} \subset \mathbb{C}$ be a finite set such that $\tilde{s} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^*$ is injective. The image is denoted by $s$.

$VB^0_\lambda (C,s) :$ Semistable bundles $E_0$ of degree 0 on $C$ such that $s(E_0) \subset s$.

$VS^\lambda (\tilde{s}) :$ Vector spaces with an endomorphism $(V,f)$ such that the eigenvalue of $f \in \tilde{s}$

The construction $(V,f) \longrightarrow (V,\frac{\partial}{\partial y} + f dy)$ gives an equivalence of categories

$VS^\lambda (\tilde{s}) \cong VB^0_\lambda (C,s)$

This equivalence will be enhanced later.

Construction $\Psi_1^\lambda$

It is convenient to consider the $C^\infty$-maps $\Psi_1 : \mathbb{A}^\lambda \longrightarrow C$ or $\Psi_1 : \mathbb{A}^\lambda \longrightarrow \mathbb{P}^1$ given by $\Psi_1(\xi,\eta) = (1 + |\lambda|^2)\Psi(\xi,\eta) = \eta + \lambda \xi^2$.

For $(\tau,y) = (\xi,\eta + \lambda \xi^2)$, we have

$\overline{\sigma}_s = \overline{\sigma}_s + \lambda \sigma_0$, $\overline{\sigma}_q = \overline{\sigma}_s$

Let $X \subset C$ be open.

$\lambda$-flat bundle on $X \Longrightarrow$ holomorphic bundle on $\Psi_1^\lambda (X)$

Let $(\mathbb{D}^\lambda (V))$ be a $\lambda$-flat bundle on $X$. A $C^\infty$-bundle $\Psi_1^\lambda (V)$ on $\Psi_1^\lambda (X)$ is equipped with an induced flat $\lambda$-connection $\mathbb{D}^\lambda (V)$ (with respect to $(\tau,y)$). Then,

$\overline{\sigma}_s = \mathbb{D}^\lambda_0 + \mathbb{D}^\lambda_1$, $\overline{\sigma}_q = \mathbb{D}^\lambda_0$

gives a holomorphic structure on $\Psi_1^\lambda (V)$. The holomorphic bundle is denoted by $\Psi_1^\lambda (V,\mathbb{D}^\lambda)$.

An analogy around infinity

$\mathbb{M}^\lambda = \mathbb{A}^\lambda \cup \mathbb{T}^\lambda$

The map $\Psi_0 : \mathbb{M}^\lambda \longrightarrow C$ is extended to a $C^\infty$-map $\Psi_0 : \mathbb{M}^\lambda \longrightarrow \mathbb{P}^1$. Let $\mathcal{X}$ be a neighbourhood of $\infty$ in $\mathbb{P}^1$.

We would like to explain the analogy between

- holomorphic vector bundles $E$ on $\Psi_0^\lambda (\mathcal{X})$ such that $E_{|\mathcal{X}}$ are semistable of degree 0,
- vector bundles $V$ on $\mathcal{X}$ with a meromorphic $\lambda$-connection $\mathbb{D}^\lambda$ such that $\mathbb{D}^\lambda (V) \subset V \otimes dw$.

(Note that $dw$ has pole of order 2 at $\infty$.)

Comparison of $\Psi_0^\lambda$ and $\Psi_1^\lambda$

$\Psi_0^\lambda$ and $\Psi_1^\lambda$ are essentially the same construction. (They are the same in the case $\lambda = 0$.)

Let $X_0 := \{ |w| > R \}$ and $X_1 := \{ |w| > (1 + |\lambda|^2)R \}$.

- We have $\Psi_0^\lambda (X_0) \cong \Psi_1^\lambda (X_1)$.
- a $\lambda$-flat bundle on $X_0$ $\longrightarrow$ a $\lambda$-flat bundle on $X_1$.

Let $(V,\mathbb{D}^\lambda)$ on $X_0$. By the parallel transport of the flat $\lambda$-connection along the segment connecting $w$ and $(1 + |\lambda|^2)w$, we obtain an isomorphism $V_{w'} \cong V_{(1 + |\lambda|^2)w}$.

It induces a $C^\infty$-isomorphism $\Psi_0^\lambda (V) \cong \Psi_1^\lambda (V)$.

We can check that it is holomorphic by an easy computation.

Extension at $\infty$.

Let $\mathcal{X} := \{ y \in \mathbb{C} | |y| \geq R \} \cup \{ \infty \}$.

Meromorphic $\lambda$-connection on $\mathcal{X} \Longrightarrow$ holomorphic vector bundle on $\Psi_1^\lambda (\mathcal{X})$

Let $V$ be a holomorphic vector bundle on $\mathcal{X}$ with a meromorphic flat $\lambda$-connection $\mathbb{D}^\lambda$ such that $\mathbb{D}^\lambda (V) \subset V \otimes dw$. The construction $\Psi_1^\lambda$ gives a holomorphic bundle $\Psi_1^\lambda (V,\mathbb{D}^\lambda)$ on $\Psi_1^\lambda (\mathcal{X})$.

Let $v_1,\ldots,v_n$ be a holomorphic frame of $V$. Let $\lambda$ be determined by $\mathbb{D}^\lambda (v_1,\ldots,v_n) = (v_1,\ldots,v_n)A(y^{-1})$, which is holomorphic in $y^{-1}$. We set $\tilde{v}_i := \Psi_1^\lambda (v_i)$.

Then,

$\overline{\sigma}_s (\tilde{v}_1,\ldots,\tilde{v}_n) = 0$, $\overline{\sigma}_q (\tilde{v}_1,\ldots,\tilde{v}_n) = (\overline{\tilde{v}_1},\ldots,\overline{\tilde{v}_n})A(y^{-1})$.

Remark $\Psi_0^\lambda$ is not naturally extended on $\mathcal{X}$. We use $\Psi_0^\lambda$ in relation with instances.
For any small sector \( S \) where \( \Sigma \) is an isomorphism, Laumon, Malgrange, Sabbah, etc. studied by Arinkin, Beilinson, Bloch, Deligne, Esnault, Fang, Fu, Graham-Squire, etc.

If we take an appropriate extension \( K \subset K_\ell \subset (\ell^\mathbb{Z}) \), we have a formal isomorphism
\[
V \otimes K_\ell \simeq \bigoplus_{a \in \ell^\mathbb{Z}} L_a \otimes R_a
\]
where \( R_a \) are regular singular, and \( L_a = C((\ell^{1/\ell}))a \) such that \( \partial_a R_a = \tau_a \partial_a a \).

The set \( \{ a \neq 0 \} \) and the formal monodromy of \( R_a \) are the important invariants for the differential module \( V \).

By the equivalence \( \text{Conn}^1(\tilde{\mathcal{X}}) \simeq \text{VB}_2(\mathcal{X}, \mathfrak{a}) \), these invariants are transferred to objects in \( \text{VB}_2(\mathcal{X}, \mathfrak{a}) \).

Formal case

Let \( \tilde{\mathcal{X}} \) denote the formal completion of \( \mathbb{P}^1 \) at \( \infty \). Let \( \mathcal{A} = \mathcal{A}^1 \) denote the formal completion of \( \mathcal{X}^1 \) along \( T^1 \). We have the formal version of the functor \( \Psi^1_\mathcal{A} \).

**Theorem** \( \Psi^1_\mathcal{A} : \text{Conn}^1(\tilde{\mathcal{X}}, \mathfrak{a}) \to \text{VB}_2^1(\mathcal{X}, \mathfrak{a}) \) is an equivalence.

It might be useful to describe the behaviour of a holomorphic vector bundle on \( \mathcal{A} \) around \( T^1 \).

"Local Fourier transform and Stationary phase formula" (Interlude)

Recall the simplest version of the generalized Fourier-Mukai transform due to Laumon-Rothstein.

Over \( T \times \mathbb{A}^1 \), we have a universal family of line bundles \( \mathcal{X} \) with a family of flat \( \lambda \)-connections \( \mathbb{A}^1 : \mathcal{X} \to \mathcal{X} \otimes \Omega^1_{T \times \mathbb{A}^1} \).

Let \( T \to T \times \mathbb{A}^1 \) be the projections.

For a meromorphic \( \lambda \)-flat bundle \( (M, \mathcal{D}^1) \) on \( T \), we obtain
\[
\text{FM}^1_\mathcal{A}(M) := \mathcal{P}^2_\mathcal{A}(M^1)(\mathcal{X}, \mathbb{A}^1)] \in \mathcal{D}^1_\mathcal{A}(\mathcal{O}_{T^1})
\]

If \( M \) is simple with \( \text{rank} M \neq 1 \), \( \text{FM}^1_\mathcal{A}(M) \) is an algebraic vector bundle on \( \mathbb{A}^1 \).

Hence, it naturally gives a locally free \( \mathcal{D}^1(\mathcal{O}_{T^1}) \)-module.

An explicit stationary phase formula for \( \text{FM}^1_\mathcal{A} \).

Let \( (M, \mathcal{D}^1) \) be a meromorphic \( \lambda \)-flat bundle on \( T \). For simplicity, we assume that \( (M, \mathcal{D}^1) \) is simple with \( \text{rank} M = 1 \). We obtain a locally free \( \mathcal{D}^1(\mathcal{O}_{T^1}) \)-module \( \text{FM}^1_\mathcal{A}(M, \mathcal{D}^1) \) on \( \mathcal{A} \).

Let \( s \subset T \) be the set of poles of \( (M, \mathcal{D}^1) \).

**Theorem**
- There exists a lattice \( E \subset \text{FM}^1_\mathcal{A}(M, \mathcal{D}^1) \) such that \( E \in \text{VB}_2^1(\mathcal{X}, \mathfrak{a}) \).
- The formal completion \( \text{FM}^1_\mathcal{A}(M, \mathcal{D}^1) \) depends only on the formal completion of \( (M, \mathcal{D}^1) \) along the poles.
- The corresponding object in \( \text{Conn}^1(\tilde{\mathcal{X}}, \mathfrak{a}) \) is described by the stationary phase formula of local Fourier transform.

Classical Hukuhara-Levelt-Turrittin decomposition

Let \( K = C((\ell^1)) \) be the field of Laurent power series. Let \( V \) be a differential \( K \)-vector space. If we take an appropriate extension \( K \subset K_\ell \subset (\ell^\mathbb{Z}) \), we have a formal isomorphism
\[
V \otimes K_\ell \simeq \bigoplus_{a \in \ell^\mathbb{Z}} L_a \otimes R_a
\]
where \( R_a \) are regular singular, and \( L_a = C((\ell^{1/\ell}))a \) such that \( \partial_a R_a = \tau_a \partial_a a \).

The set \( \{ a \neq 0 \} \) and the formal monodromy of \( R_a \) are the important invariants for the differential module \( V \).

By the equivalence \( \text{Conn}^1(\tilde{\mathcal{X}}) \simeq \text{VB}_2(\mathcal{X}, \mathfrak{a}) \), these invariants are transferred to objects in \( \text{VB}_2(\mathcal{X}, \mathfrak{a}) \).

Classical Fourier transform

We have a line bundle with a flat connection \( (\mathcal{E} \subset \mathcal{C}, \mathcal{D} + d(\mathcal{L})) \) on \( \mathcal{C} \times \mathcal{C} \).

For a meromorphic flat bundle \( (M, \mathcal{D}^1) \) on \( \mathcal{C} \), we have
\[
\mathfrak{F}(M, \mathcal{D}^1) := \mathcal{P}^2_\mathfrak{C}(M) \otimes (\mathcal{C}, \mathcal{D}^1)] \in \mathcal{D}^1_\mathfrak{C}(\mathfrak{C})
\]

For \( \mathfrak{e} \), a local Fourier transform and an explicit stationary phase formula were studied by Arinkin, Bellinison, Bloch, Deligne, Esnault, Fang, Fu, Graham-Squire, Laumon, Malgrange, Sabbah, etc.

The corresponding object in \( \text{Conn}^1(\tilde{\mathcal{X}}) \) is the equivalence \( \text{Conn}^1(\tilde{\mathcal{X}}) \simeq \text{VB}_2(\mathcal{X}, \mathfrak{a}) \).

Asymptotic analysis

We come back to the study of \( E \in \text{VB}_2^1(\mathcal{X}, \mathfrak{a}) \), where \( X = \{ y \in C \mid |y| \geq R \} \).

\( X = X \cup \{ \infty \} \) and \( \mathcal{A} = \mathcal{A}^1(\mathfrak{C}) \).

There exists \( (V, \mathcal{D}^1) \in \text{Conn}^1(\tilde{\mathcal{X}}, \mathfrak{a}) \) such that
\[
\Psi^1_\mathcal{A}(V, \mathcal{D}^1) \simeq E \mathcal{D}^1.
\]

(1)

As in the case of meromorphic flat bundles, the isomorphism is not convergent, in general.

**Theorem** For any small sector \( S \subset X \), there exists a holomorphic isomorphism \( E\Psi^1_\mathcal{A}(S) \simeq \Psi^1_\mathcal{A}(V, \mathcal{D}^1)\Psi^1_\mathcal{A}(S) \), asymptotic to (1).

(1) is an admissible trivialization in this talk.

A sector is \( S = \{ w \in C \mid |w| \geq R, \theta_0 \leq \arg(w) \leq \theta_1 \} \).

This is an analogue of the classical asymptotic analysis for meromorphic flat bundles.
For simplicity, we assume $\{V, D^\beta\} = \oplus_{\beta \in \mathbb{C}} (V, D^\beta)$ for $(V, D^\beta) \in \mathbb{C}^{1}(\mathcal{X}, \partial)$. Let $v_1, \ldots, v_\delta$ be a frame of $V_\alpha$, obtained from frames of $V_{\beta}$, $v_1 \in V_{\beta}$. Let $U \subset S$ be any open subset. A $C^\infty$-section $f$ of $\Psi^1(V, D^\beta)$ on $\Psi^1(U)$ is expressed as

$$f = \sum_{\beta} f_\beta \varphi_\beta(d^\beta y).$$

We set $\mathcal{F}^{(j)}(\Psi^1(V)(U)) := \{ f : \mathcal{F} \rightarrow \mathbb{C} \}$ with $\mathcal{F}$ being a field.

We define a filtration $\mathcal{F}^{(j)}(\Psi^1(E)(\beta))$ by using an admissible trivialization.

**Proposition**

- The filtration is independent of the choice of an admissible trivialization. It is characterized in terms of the growth order.
- The filtration is preserved by the $\lambda$-connection.
- For $S \subset X$, we have $\mathcal{F}^{(j)}(\Psi^1(V)(\beta)) \subset \mathcal{F}^{(j)}(\Psi^1(E)(\beta))$.
- We put $\mathcal{F}^{(j)}(\Psi^1(E)(\beta)) = \mathcal{F}^{(j)}(\Psi^1(V)(\beta)) \mathcal{F}^{(j)}(\Psi^1(E)(\beta))$.

We obtain a function $\mathcal{G}^{(j)}(\Psi^1(E)_\beta) : \mathbb{R}^{\mathbb{C}}(\mathbb{C}) \rightarrow \mathcal{F}^{(j)}(\mathbb{C}, \partial \mathcal{X})$ for $\alpha = \partial \beta + \lambda \beta$.

$\mathcal{G}^{(j)}(\Psi^1(E)_\beta)$ may have non-trivial Stokes structure. It is not necessarily isomorphic to $(V, D^\beta)$.

We have a similar classical construction $\mathcal{G}^{(j)} : \mathbb{R}^{\mathbb{C}}(\mathbb{C}) \rightarrow \mathcal{F}^{(j)}(\mathbb{C}, \partial \mathcal{X})$ for $\alpha = \partial \beta$. We have $\mathcal{G}^{(j)}(\Psi^1(E)_\beta) = \mathcal{G}^{(j)}(\Psi^1(E)_\beta)$.
Application to instantons on $T^\vee \times \mathbb{C}$

**Instanton**

We use the metric $dz \, dz + dw \, dw$ on $T^\vee \times \mathbb{C}$. Let $X := \{ w \in \mathbb{C} \mid |w| \geq R \}$. Let $E$ be a $C^\infty$-bundle on $\Psi_0^n(X) = T^\vee \times X$ with a hermitian metric $h$ and a unitary connection $\nabla$. The curvature of $\nabla$ is denoted by $F(\nabla)$.

The connection $\nabla$ is called self-dual, if $-F(\nabla) = F(\nabla)$, where $\ast$ denotes the Hodge star operator. In this case, $(E,\nabla,h)$ is called an instanton.

It is equivalent to the following:

- The $(0,1)$-part of $\nabla$ gives a holomorphic structure.
- For the expression $F(\nabla) = F_3 dz \wedge dw + F_0 (dz \wedge dw) + F_{1\bar{1}} dz \wedge d\bar{w} + F_{00} dw \wedge d\bar{w}$, we have $F_{\bar{3}} + F_{3\bar{3}} = 0$.

We would like to explain how to use the Stokes structure of vector bundles on $T^\vee \times X$ for the study of instantons on $\mathbb{A}^2$ such that $F(\nabla)$ is $L^2$.

**Nahm transform**

For a closed subgroup $\Gamma \subset \mathbb{R}^3$, let $\Gamma^\vee := \{ \chi \in (\mathbb{R}^3)^\vee \mid \chi(\Gamma) \subset \mathbb{Z} \}$.

It is believed and established in some degree

\[
\begin{pmatrix}
\text{\Gamma-equivariant instanton} \\
\text{satisfying some condition} \\
\text{with some singularity}
\end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix}
\text{\Gamma'-equivariant instanton} \\
\text{satisfying some condition} \\
\text{with some singularity}
\end{pmatrix}
\]

An instanton on $T^\vee \times \mathbb{C}$ is $\Gamma^\vee$-equivariant instanton.

- ADHM construction (Atiyah-Drinfeld-Hitchin-Manin) in the case $\Gamma = \{ 1 \}$ and $\Gamma^\vee = \mathbb{R}^3$.
- Nahm studied the case $\Gamma = \mathbb{R}$ and $\Gamma^\vee = \mathbb{R}^3$. It was refined by Hitchin and Nakajima.

Since then, the other cases were also studied by many people.

**Harmonic bundle**

Let $(E,\nabla,h)$ be an instanton on $T^\vee \times X$ which is $T^\vee$-equivariant.

- We obtain a $C^\infty$-bundle $E_1$ on $X$ with a hermitian metric $h_1$ such that $\Psi_0(E_1,h_1) = (E,h)$.
- We also have a unitary connection $\nabla_1$ of $(E_1,h_1)$ such that $\Psi_0(\nabla_1)(v) = \nabla(v)$ for $v = a \delta_i + b \psi_i$.
- Because $\nabla$ is $T^\vee$-equivariant, $\nabla - \Psi_0(\nabla_1) = \nabla_1 + f_d$ for $f \in \text{End}(E_1)$.

The anti-self duality condition is reduced to the Hitchin equation

$$ F(\nabla_1) + [f dw, f'] d\bar{\sigma} = 0 $$

$(E_1,\nabla_1, f dw)$ with the metric $h$ called a harmonic bundle, where $\nabla_1$ is the $(0,1)$-part of $\nabla$.

**Hitchin**

$T^\vee$-equivariant instanton on $T^\vee \times X$ is equivalent to a harmonic bundle on $X$.

**What I would like to do?**

The case $\Gamma = \mathbb{A}^1$ and $\Gamma^\vee = \mathbb{A} \times \mathbb{C}^2$ was previously studied by Jardim collaborated with Biquard. They established the Nahm transform between

- Harmonic bundles on $T$ with tame singularity.
- Instantons on $T^\vee \times X$ satisfying the quadratic decay condition. i.e., $|F(\nabla)| = O(|w|^{-2})$ with respect to $h$ and $dz \wedge dw$.

**My goals**

1. Refine the condition from “quadratic decay” to “$L^2$”, and establish the correspondence between
   - Harmonic bundles on $T$ with wild singularity
   - Instantons on $T^\vee \times X$ such that $F(\nabla)$ is $L^2$.
   (We do not explain this anymore in this talk.)
2. Refine the study by using the twistor viewpoint.
   - Stokes structure naturally appears.
   - We obtain wild harmonic bundle as a graduation of instanton with respect to the Stokes structure.

Let $\mathbb{C} \times \mathfrak{T} \longrightarrow T \times \mathfrak{T}$ be the morphism induced by a universal covering $\mathbb{C} \longrightarrow T$. We fix a lift $\mathfrak{X} \subset \mathbb{C} \times \mathfrak{T}$ of $\mathfrak{T}$, and put $z := \pi(\mathfrak{X} \cap \{ \{w\} \})$.

**Lemma**

$\exists R > 0$ such that $(E,\nabla)_{|T^\vee \times \{w\}}$ are semistable of degree $0$ for any $w \in X$ with $|w| > R$.

We may assume that $(E,\nabla)_{|T^\vee \times \{w\}}$ are semistable of degree $0$ from the beginning.

By the relative Fourier-Mukai transform, we obtain a coherent sheaf $FM(E)$ on $T \times X$. The support $\mathfrak{X} \subset T \times X$ is relatively $0$-dimensional over $X$.

**Proposition**

$\mathfrak{X}$ is naturally extended to a subvariety $\mathfrak{X}$ in $T \times \mathfrak{T}$.

Let $\mathfrak{X} \subset \mathfrak{T}$ be a morphism induced by a universal covering $\mathbb{C} \longrightarrow T$. We fix a lift $\mathfrak{X} \subset \mathbb{C} \times \mathfrak{T}$ of $\mathfrak{T}$, and put $z := \pi(\mathfrak{X} \cap \{ \{w\} \})$.

**Lemma**

$\exists (V^0,D^0) \in \text{Conn}(\mathfrak{T},\mathfrak{T})$ such that $\Psi_0(V^0,D^0) = (E,\nabla)$.

We obtain the following theorem.

**Theorem**

We have an induced harmonic metric $h_0$ of $(V^0,D^0)$, for which

$$ \Psi_0(h_0) - h = O(\exp(-C|w|^2)) $$

for some $C,\delta > 0$.

We would like to explain how to obtain a harmonic bundle $(V^0,D^0,h_0)$, or equivalently $T^\vee$-equivariant instanton $\Psi_0(V^0,D^0,h_0)$, by using the previous consideration on the Stokes structure of objects in $\text{VB}_0(\mathfrak{T})$. 
Deligne-Hitchin space
We recall the construction of Deligne-Hitchin space

- We have the natural family $\mathcal{M} \rightarrow \mathbb{C}$ such that the fiber $\mathcal{M} \times \mathbb{C} \{\lambda\}$ is $\mathcal{M}_\lambda$.
- We also have the natural family $\mathcal{M}^\dagger \rightarrow \mathbb{C}$ such that the fiber $\mathcal{M}^\dagger \times \mathbb{C} \{\mu\}$ is the moduli of line bundles with flat $\mu$-connection on $T'$, where $T'$ denotes the conjugate of $T$.
- We have the natural holomorphic isomorphism $\mathcal{M} \times \mathbb{C} \cong \mathcal{M}^\dagger \times \mathbb{C}$.
- $\lambda^{-1} = \mu$.
- By gluing, we obtain a complex manifold $\mathcal{M}_{DH}$ with a morphism $\mathcal{M}_{DH} \rightarrow \mathbb{P}^1_\mathbf{C}$. (The twistor space of the hyperkähler manifold $T' \times \mathbb{C}$.)

We recall some basic facts.

- We have a $C^\infty$-isomorphism $\mathcal{M}_{DH} \cong \mathbb{P}^1_\mathbf{C} \times T' \times \mathbb{C}$.
- The twistor lines $C_Q := \mathbb{P}^1_q \times \{Q\}$ are complex submanifolds for any $Q \in T' \times \mathbb{C}$.
- We have an anti-holomorphic involution $\sigma : \mathcal{M}_{DH} \rightarrow \mathcal{M}_{DH}$, compatible with $\sigma : \mathbb{P}^1_\mathbf{C} \rightarrow \mathbb{P}^1_\mathbf{C}$ given by $\sigma(\lambda) = -\lambda^{-1}$.

Twistor description of an instanton

- We have the $C^\infty$-map $\Psi_{DH} : \mathcal{M}_{DH} = \mathbb{P}^1_\mathbf{C} \times T' \times \mathbb{C} \rightarrow \mathbb{C}$.
- For $X = \{w \in \mathbb{C} | |w| \geq R\}$, we set $\mathcal{X}_{DH} = \Psi_{DH}(X)$.

Recall the following well known fact.

An instanton on $T' \times X$ is equivalent to a holomorphic vector bundle $\mathcal{E}_{DH}$ on $\mathcal{X}_{DH}$ with a holomorphic pairing $P : \mathcal{E}_{DH} \times \sigma^* \mathcal{E}_{DH} \rightarrow \mathcal{E}_{DH}$ satisfying the following for any $Q \in T' \times X$.

- $(\mathcal{E}_{DH}, P_Q)$ are polarized pure twistor structure of weight 0, i.e., $\mathcal{E}_{DH,Q} : \mathcal{E}_{DH,Q}$ are isomorphic to $\mathcal{O}(n)$, and $P_Q$ induces a positive definite hermitian metric of $H^0(\mathcal{O}(Q))$.

Prolongation

Let $(E, h, \nabla)$ be an $L^2$-instanton on $T' \times X$. Let $\mathcal{E}$ be the corresponding holomorphic vector bundle on $\mathcal{X}_{DH}$. For $\lambda \in \mathbb{P}^1_\mathbf{C}$, we set $\mathcal{E}_\lambda := \mathcal{E}_\mathbb{C}$.

**Proposition** $(\mathcal{E}_\lambda, h)$ is acceptable, i.e., the curvature of $(\mathcal{E}_\lambda, h)$ is bounded with respect to $h$ and the Poincaré like metric of $\mathcal{E}_\lambda$. For each $a \in \mathbb{R}$, we obtain an $\mathcal{E}_\lambda$-module $\mathcal{P}_a\mathcal{E}_\lambda$ such that $\mathcal{P}_a\mathcal{E}_{a+1}^\lambda = \mathcal{E}_\lambda$.

**Proposition** $\mathcal{P}_a\mathcal{E}_\lambda$ is an object in $\mathcal{V}_{\infty}(\mathcal{X}_{DH})$.

Taking Gr

We obtain a vector bundle with a meromorphic flat $\lambda$-connection on $\Psi_{\infty}(\mathcal{X}_{\lambda})$.

**Proposition** $\bigcup_{a \in \mathbb{R}} \mathcal{P}_a\mathcal{E}_\lambda$ naturally gives a holomorphic vector bundle $\mathcal{E}_\lambda$ on $\mathcal{X}_{DH} \cap \mathcal{X}$. (Recall $\mathcal{M}_{DH} = \mathcal{M} \cup \mathcal{M}^\dagger$.)

- By considering the conjugate, we obtain $\mathcal{E}_\lambda$ on $\mathcal{X}_{DH} \cap \mathcal{X}^\dagger$ over $\mathbb{P}^1_\mathbf{C} \setminus \{0\}$.
- We have a natural isomorphism $\mathcal{E}_\lambda \cap \mathcal{X}^\dagger \cong \mathcal{E}_\lambda \cap \mathcal{X} \cap \mathcal{X}^\dagger$.
- By gluing, $\mathcal{E}_\lambda$ and $\mathcal{E}_\lambda^\dagger$, we obtain a holomorphic vector bundle $\mathcal{E}_{DH}$ on $\mathcal{X}_{DH}$.
- We have a naturally induced pairing $P : \mathcal{E}_{DH} \times \sigma^* \mathcal{E}_{DH} \rightarrow \mathcal{E}_{DH}$.

**Theorem** After $X$ is shrank appropriately, $(\mathcal{E}_{DH}, P)$ gives an instanton. It is $T'$-equivariant.