Star-exponentials on a complex symplectic manifold
(joint work with Pierre Schapira)

Giuseppe Dito

Institut de Mathématiques de Bourgogne
UMR CNRS 5584
Université de Bourgogne
Dijon, France

Spring School on Algebraic Microlocal Analysis
Northwestern University, May 14, 2012
Introduction and motivations

- Given a star-product $\star_\hbar$ on a symplectic manifold $M$, the star-exponential of $H: M \to \mathbb{R}$ is defined by the series:

$$\text{Exp}_{\star_\hbar}(\frac{tH}{i\hbar}) = \sum_{n \geq 0} \frac{1}{n!} \left(\frac{t}{i\hbar}\right)^n H_{\star_\hbar}^n$$

(Here $\hbar$ is not a formal parameter but a positive real number.)

- It was introduced in [BFFLS 1978] as a tool to study the spectrum of observables without referring to an underlying Hilbert space.

- If $\hbar$ is a formal parameter, the star-exponential does not have any obvious meaning in the deformation quantization algebra $C^\infty(M)[[\hbar]]$. It would rather belong to $C^\infty(M)[[\hbar, \hbar^{-1}]]$.

- But $(C^\infty(M)[[\hbar, \hbar^{-1}]], \star_\hbar)$ is not an algebra.

- With P. Schapira, by using techniques from micolocal analysis, we have constructed an algebra of deformation quantization on the cotangent bundle of a complex manifold, containing the star-exponentials.
Example (Harmonic oscillator (BFFLS Ann. Phys. 1978))

\( T^* \mathbb{R} = \mathbb{R}^2 \) with Moyal product \( \star_M \). Hamiltonian: 
\[
H(x, \xi) = \frac{1}{2}(\xi^2 + x^2)
\]

\[
\text{Exp}_{\star_M} \left( \frac{tH}{i\hbar} \right) = \frac{1}{\cos(t/2)} \exp \left( \frac{(x^2 + \xi^2)}{i\hbar} \tan(t/2) \right)
\]

for \( |t| < \pi \). The convergence is in \( \mathcal{D}'(\mathbb{R}^2) \). It is a periodic distribution in \( t \).

\[
\frac{1}{\cos(t/2)} \exp \left( \frac{(x^2 + \xi^2)}{i\hbar} \tan(t/2) \right) = \sum_{n \geq 0} \exp(-i(n + \frac{1}{2})t)\pi_n(x, \xi)
\]

where

\[
\pi_n(x, \xi) = 2(-1)^n \exp(-\frac{(x^2 + \xi^2)}{\hbar})L_n\left( \frac{2(x^2 + \xi^2)}{\hbar} \right),
\]

where the \( L_n \)'s are the Laguerre polynomials.

\[
H \star_M \pi_n = \hbar(n + 1/2)\pi_n \quad \pi_n \star_M \pi_{n'} = \delta_{nn'}\pi_n.
\]
Example (Feynman Path Integral (GD LMP 1990))

- Normal star-product:

\[(f \star_N g)(\bar{z}, z) = fg + \sum_{n \geq 1} \frac{\hbar^n}{n!} \frac{\partial^n f}{\partial z^n} \frac{\partial^n g}{\partial \bar{z}^n} \cdot\]

- In the holomorphic representation of the CCR \([a, a^\dagger] = \hbar\]

\[(af)(\bar{z}) = \hbar f'(\bar{z}) \quad (a^\dagger f)(\bar{z}) = \bar{z} f(\bar{z})\]

the FPI takes the form (Faddeev, Les Houches, 1975):

\[
\int \prod_s \frac{d\bar{\xi}_s d\xi_s}{2\pi i \hbar} \exp \left[ \frac{1}{2} (\bar{z} \xi_t + z \bar{\xi}_0) - \frac{1}{\hbar} \int_0^t ds \frac{1}{2} (\bar{\xi}_s \dot{\xi}_s - \dot{\bar{\xi}}_s \xi_s) + H(\bar{\xi}_s, \xi_s) \right]
\]

integration is over paths \(s \mapsto (\bar{\xi}_s, \xi_s)\) restricted to boundary conditions \(\bar{\xi}_t = \bar{z}\) and \(\xi_0 = z\).

- Heuristically:

"\(\text{Exp}_{\star_N} \left( \frac{tH}{i\hbar} \right)(\bar{z}, z) = \exp(-\frac{1}{\hbar} \bar{z} z) \cdot \text{FPI}(t, H)(\bar{z}, z)"\)
The sheaf of microdifferential operators $\mathcal{E}_{T^*X}$

- Let $X$ be a complex manifold.
- At the beginning of the 70’s, Sato-Kashiwara-Kawai (and Louis Boutet de Monvel) have constructed the sheaf of microdifferential operators $\mathcal{E}_{T^*X}$.
- $\mathcal{E}_{T^*X}$ is a $C^\infty$-conic filtered sheaf of rings.
- Locally: $U \subset T^*X$, $(x, \xi) \in U$, a section $P \in \mathcal{E}_{T^*X}(U)$ is described by its total symbol:

$$
\sigma_{\text{tot}}(P)(x; \xi) = \sum_{-\infty < j \leq m} p_j(x; \xi), \quad m \in \mathbb{Z}, \quad p_j \in \Gamma(U; \mathcal{O}_{T^*X}(j)).
$$

- $\sigma_{\text{tot}}(P)$ satisfies growth conditions (canonical estimates):

$$
\left\{ \begin{array}{l}
\text{for any compact subset } K \text{ of } U \text{ there exist positive constants } C, \varepsilon \text{ such that } \\
\sup_{(x;\xi) \in K} |p_j(x; \xi)| \leq C \varepsilon^{-j}(-j)! \text{ for all } j < 0.
\end{array} \right.
$$

- If $Q$ is an operator of total symbol $\sigma_{\text{tot}}(Q)$, then the total symbol of the product $P \circ Q$ is given by the Leibniz product:

$$
\sigma_{\text{tot}}(P \circ Q) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{\text{tot}}(P) \partial_x^\alpha \sigma_{\text{tot}}(Q).
$$
The sheaf of microdifferential operators $\mathcal{E}_{T^*X}$

- Filtered by the order of operators: $\mathcal{E}_{T^*X} = \bigcup_{m \in \mathbb{Z}} \mathcal{E}_{T^*X}(m)$
- The associated graded sheaf of rings

$$\text{gr} \mathcal{E}_{T^*X} \cong \bigoplus_{j \in \mathbb{Z}} \mathcal{O}_{T^*X}(j).$$

- Consider a *homogeneous* symplectic transformation

$$\varphi: T^*X \supset U \xrightarrow{\sim} V \subset T^*Y.$$  

Then $\varphi$ may be locally quantized as an isomorphism of filtered sheaves of rings

$$\Phi: \varphi_*\mathcal{E}_{T^*X}|_U \xrightarrow{\sim} \mathcal{E}_{T^*Y}|_V.$$  

Remark: This isomorphism exists locally and is not unique.
Set \( \hat{k} = \mathbb{C}[[\hbar, \hbar^{-1}]] \). An element \( a \in \hat{k} \) of order \( m \in \mathbb{Z} \) is a formal series:

\[
a = \sum_{-\infty < j \leq m} a_j \hbar^{-j}, \quad a_j \in \mathbb{C}.
\]

One defines \( k \) as the subfield of \( \hat{k} \) of series satisfying

there exist \( C, \varepsilon > 0 \) with \( |a_j| \leq C \varepsilon^{-j}(-j)! \) for all \( j < 0 \).
The sheaf $\mathcal{W}_{T^*X}$

- If one forgets about the homogeneity of $T^*X$, there exists a no more conic filtered sheaf of $k$-algebras $\mathcal{W}_{T^*X}$.
- It is a special case of a more general construction (algebroid stacks) of deformation quantization of a complex symplectic manifold. [Kontsevich (LMP 2001) for the formal case, Polesello-Schapira (IMRN 2004) for the analytic case in the spirit of the construction by Kashiwara (1996) of the quantization for complex contact manifolds.]
- Introduce a new parameter $\hbar$ to replace homogeneity.
- The formal version of $\mathcal{W}_{T^*X}$ is similar to deformation quantization in the $C^\infty$ setting.
The sheaf $\mathcal{W}_{T^*X}$

- Locally $\mathcal{W}_{T^*X}$ is described as follows. $U \subset T^*X$, a section $P \in \mathcal{W}_{T^*X}(U)$ has a total symbol

$$\sigma_{tot}(P)(x; \xi) = \sum_{-\infty < j \leq m} p_j(x; \xi)\hbar^{-j}, \quad m \in \mathbb{Z}, \quad p_j \in \mathcal{O}_{T^*X}(U),$$

for any compact subset $K$ of $U$ there exist constants $C, \varepsilon > 0$ such that

$$\sup_{(x;\xi) \in K} |p_j(x, \xi)| \leq C\varepsilon^{-j}(-j)! \text{ for all } j < 0.$$  

- Its associated graded ring is

$$\text{gr } \mathcal{W}_{T^*X} \simeq \mathcal{O}_{T^*X}[\hbar, \hbar^{-1}].$$

- The product is given by the Leibniz product:

$$\sigma_{tot}(P) \ast \sigma_{tot}(Q) := \sum_{\alpha \in \mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!} \partial_{\xi}^\alpha \sigma_{tot}(P) \cdot \partial_x^\alpha \sigma_{tot}(Q).$$
The sheaf $\mathcal{W}_{T^*X}$

- A symplectic transformation

$$\varphi: T^*X \supset U \xrightarrow{\sim} V \subset T^*Y.$$ 

as in the case of $E_{T^*X}$, $\varphi$ can be locally quantized as an isomorphism of filtered sheaves of rings

$$\Phi: \varphi_* \mathcal{W}_{T^*X}|_U \xrightarrow{\sim} \mathcal{W}_{T^*Y}|_V.$$ 

Again, this isomorphism exists locally and is not unique.
From $\mathcal{E}_{T^*X}$ to $\mathcal{W}_{T^*X}$

The sheaves $\mathcal{E}_{T^*X}$ and $\mathcal{W}_{T^*X}$ are linked as follows. Let $t \in \mathbb{C}$ be the coordinate and define

$$\mathcal{E}_{T^* (X \times \mathbb{C})}, \hat{t} = \{ P \in \mathcal{E}_{T^* (X \times \mathbb{C})}; [P, \partial_t] = 0 \}. $$

Set

$$T^*_{\tau \neq 0} (X \times \mathbb{C}) = \{ (x, t; \xi, \tau); \tau \neq 0 \}$$

and consider the map

$$\rho: T^*_{\tau \neq 0} (X \times \mathbb{C}) \to T^* X, \quad \rho(x, t; \xi, \tau) = (x; \xi/\tau).$$

The ring $\mathcal{W}_{T^*X}$ on $T^* X$ is given by

$$\mathcal{W}_{T^*X} := \rho_* (\mathcal{E}_{T^* (X \times \mathbb{C})}, \hat{t} | _{T^*_{\tau \neq 0} (X \times \mathbb{C})}).$$

(One should think of $\tau$ as being $\hbar^{-1}$.)
commutative \[ \mathcal{O}_{X}^{s,\hbar} \] \[ \leadsto \] \[ \mathcal{W}_{T^*X}^{s} \] (resolvent) noncommutative \[ \mathcal{O}_{X}^{t,\hbar} \] \[ \leadsto \] \[ \mathcal{W}_{T^*X}^{t} \] (exponential)

\[ \frac{1}{s-H} \in \mathcal{W}_{T^*X}^{s} \]

\[ \exp\left(\frac{tH}{\hbar}\right) \in \mathcal{W}_{T^*X}^{t} \]

\[ \frac{\partial}{\partial t} \Phi(t) = \frac{1}{\hbar} H\Phi(t), \quad \Phi(0) = 1 \]
The sheaf $\mathcal{O}_X^{s,\hbar}$

**Definition ($\mathcal{O}_X^{\hbar}$)**

We denote by $\mathcal{O}_X^{\hbar}$ the filtered sheaf of $k$-algebras whose sections of order $m$ on an open set $U \subset X$ are series

$$f(x, \hbar) = \sum_{-\infty < j \leq m} f_j(x)\hbar^{-j}, \quad f_j \in \mathcal{O}_X(U),$$

satisfying:

$$\begin{cases} 
\text{for any compact subset } K \text{ of } U \text{ there exist positive constants } C, \varepsilon \text{ such that } \sup_K |f_j| \leq C\varepsilon^{-j}(-j)! \text{ for all } j < 0. 
\end{cases}$$

Let $\mathbb{C}_s$ denote $\mathbb{C}$ with coordinate $s$. Let $a: \mathbb{C}_s \times X \to X$ be the projection. The sheaf $\mathcal{O}_X^{s,\hbar}$ is defined as the derived proper direct image:

$\mathcal{O}_X^{s,\hbar} := R^1a_!\mathcal{O}_R^{\hbar}$
The sheaf $\mathcal{O}_{X}^{s, \hbar}$ — The convolution algebra $H^{1}_{c}(\mathbb{C}; \mathcal{O}_{\mathbb{C}})$

- For a compact subset $K$ of $\mathbb{C}$, we identify the vector space $H^{1}_{K}(\mathbb{C}; \mathcal{O}_{\mathbb{C}})$ with the quotient space $\Gamma(\mathbb{C} \setminus K; \mathcal{O}_{\mathbb{C}})/\Gamma(\mathbb{C}; \mathcal{O}_{\mathbb{C}})$ and, if $f \in \Gamma(\mathbb{C} \setminus K; \mathcal{O}_{\mathbb{C}})$, we still denote by $f$ its image in $H^{1}_{K}(\mathbb{C}; \mathcal{O}_{\mathbb{C}})$ or in $H^{1}_{c}(\mathbb{C}; \mathcal{O}_{\mathbb{C}})$.

- Let $K$ and $L$ be compact subsets of $\mathbb{C}$, let $f \in \Gamma(\mathbb{C} \setminus K; \mathcal{O}_{\mathbb{C}})$ and $g \in \Gamma(\mathbb{C} \setminus L; \mathcal{O}_{\mathbb{C}})$.

- The convolution product $f *_{c} g$ is given by

$$
(f *_{c} g)(z) = \frac{1}{2i\pi} \int_{\gamma} f(z - w) g(w) dw
$$

(1)

where $\gamma$ is a counter clockwise oriented circle which contains $L$ and $|z|$ is chosen big enough so that $z + K$ is outside of the disc bounded by $\gamma$.

- $(H^{1}_{c}(\mathbb{C}; \mathcal{O}_{\mathbb{C}}), *_{c})$ is an abelian algebra.

- 

$$
\frac{1}{z^{n+1}} *_{c} \frac{1}{z^{m+1}} = \frac{(n + m)!}{n!m!} \frac{1}{z^{n+m+1}}.
$$
The sheaf $\mathcal{O}_{\bar{X}}^{s, \hbar}$

- For $U$ open, relatively compact in $X$, sections of order $m$ defined on a neighborhood of $\overline{U}$, are described by:

$$f(s, x, \hbar) = \sum_{j \leq m} f_j(s, x) \hbar^{-j},$$

- $f_j(s, x)$ is holomorphic on $(\mathbb{C}_s \setminus K_0) \times U$, $K_0$ compact independent of $j$.
- $\forall K \subset (\mathbb{C}_s \setminus K_0) \times U$, we have canonical estimates.
- $\mathcal{O}_{\bar{X}}^{s, \hbar}$ is a sheaf of filtered $k$-modules.
- Extend the convolution product to $\mathcal{O}_{\bar{X}}^{s, \hbar}$ as follows. For two sections $f(s, x, \hbar) = \sum_{-\infty < j \leq m} f_j(s, x) \hbar^{-j}$ and $g(s, x, \hbar) = \sum_{-\infty < j \leq m'} g_j(s, x) \hbar^{-j}$ of $\mathcal{O}_{\bar{X}}^{s, \hbar}$, set:

$$\begin{cases} f(s, x, \hbar) \ast_c g(s, x, \hbar) = \sum_{-\infty < j \leq m + m'} h_j(s, x) \hbar^{-j}, \\ h_k(s, x) = \sum_{i+j=k} \frac{1}{2i\pi} \int_\gamma f_i(s - w, x) g_j(w, x)dw. \end{cases}$$

**Theorem**

The sheaf $\mathcal{O}_{\bar{X}}^{s, \hbar}$ has a structure of a filtered abelian $k$-algebra.
The sheaf $\mathcal{O}^{t,\hbar}_X$

- Locally, sections of order $m$ are described by:

$$U \subset X, \quad f(t, x, \hbar) = \sum_{j \in \mathbb{Z}} f_j(t, x) \hbar^{-j}, \quad f_j \in \Gamma(U, \mathcal{O}_{\mathbb{C} t \times X}|_{t=0})$$

- $\forall K \subset U$, $\exists \eta > 0$:
  - $f_j(t, x)$ is holomorphic around $\{|t| \leq \eta\} \times K$
  - $\exists C, \varepsilon > 0, \sup_{x \in K, |t| \leq \eta} |f_j(t, x)| \leq C \varepsilon^{-j}(-j)!$ for all $j < 0$.
  - $\exists M, R > 0, \sup_{x \in K} |f_j(t, x)| \leq M \frac{R^{j-m}}{(j-m)!} |t|^{j-m}, \quad \forall |t| \leq \eta \ \forall j \geq m.$
The sheaf $\mathcal{O}_{\mathcal{X}}^{t,\hbar}$

Facts:

- $\hbar^{-1}: \mathcal{O}_{\mathcal{X}}^{t,\hbar}(m) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}}^{t,\hbar}(m + 1)$.
- If $f \in \mathcal{O}_{\mathcal{X}}^{t,\hbar}(m)$ and $g \in \mathcal{O}_{\mathcal{X}}^{t,\hbar}(m')$, then $fg \in \mathcal{O}_{\mathcal{X}}^{t,\hbar}(m + m')$.

Theorem

$\mathcal{O}_{\mathcal{X}}^{t,\hbar}$ is a sheaf of abelian filtered $k$-algebras.

- The sheaf $\mathcal{O}_{\mathcal{X}}^{t,\hbar}$ does not admit a formal counterpart.
Laplace transform

- The sheaves $\mathcal{O}_X^{t,\hbar}$ and $\mathcal{O}_X^{s,\hbar}$ are related by a kind of Laplace transform.
- On an open set $U$ of $X$, consider a section:

$$f(s, x, \hbar) \in \Gamma((\mathbb{C}_s \setminus K) \times U; \mathcal{O}_{\mathbb{C}_s \times X}^\hbar).$$

i.e.

$$f(s, x, \hbar) = \sum_{-\infty < j \leq m} f_j(s, x) \hbar^{-j},$$

- Define the Laplace transform $\mathcal{L}(f)$ of $f$ by

$$\mathcal{L}(f)(t, x, \hbar) = \frac{1}{2i\pi} \int_\gamma f(s, x, \hbar) \exp(st\hbar^{-1}) \, ds,$$

where $\gamma$ is a counter clockwise oriented circle centered at 0 with radius $R \gg 0$.

- 

$$\mathcal{L}(s^{-n-1}) = \hbar^{-n}t^n/n!, \quad \mathcal{L}\left(\frac{1}{s-1}\right) = \exp(t\hbar^{-1}).$$
Theorem

The Laplace transform induces a \( k \)-linear isomorphism of filtered \( k \)-algebras

\[
\mathcal{L} : \mathcal{O}^{s,\hbar}_X \sim \rightarrow \mathcal{O}^{t,\hbar}_X.
\]

▶ Note: A formal version \( \hat{\mathcal{O}}^{s,\hbar}_X \) of \( \mathcal{O}^{s,\hbar}_X \) does exist, but the Laplace transform cannot be applied to that formal version.
▶ Take a sequence \( \{c_j\}_{j \leq 0} \) in \( \mathbb{C} \) and consider the section \( f \) of \( \hat{\mathcal{O}}^{s,\hbar}_X \):

\[
f(s, \hbar) = \sum_{j \leq 0} \frac{c_j}{(s - 1)} \hbar^{-j}.
\]

▶ Then, formally, the Laplace transform of \( f \) is given by

\[
\mathcal{L}(f)(t, \hbar) = \sum_{j \leq 0} \sum_{n \geq 0} c_j \frac{t^n}{n!} \hbar^{-n-j},
\]

▶ The coefficient of \( \hbar^0 \) is \( \sum_{n \geq 0} c_n \frac{t^n}{n!} \), which does not converge around \( t = 0 \) in general.
The sheaf $\mathcal{W}^s_{T^*X}$

Denote by $s$ the coordinate on $\mathbb{C}_s$. Let $\mathcal{W}_{\mathbb{C}_s \times T^*X}$ be the subsheaf of $\mathcal{W}_{T^*(\mathbb{C} \times X)}$ consisting of sections not depending on $\partial_s$:

$$\mathcal{W}_{\mathbb{C}_s \times T^*X} = \{ P \in \mathcal{W}_{T^*(\mathbb{C}_s \times X)} \mid [P, s] = 0 \}.$$  

As for $\mathcal{O}_{X}^{s, \hbar}$, the sheaf $\mathcal{W}^s_{T^*X}$ is defined as a proper direct image of $\mathcal{W}_{\mathbb{C}_s \times T^*X}$ by the projection $a: \mathbb{C}_s \times X \to X$:

**Definition**

The sheaf of $\mathbf{k}$-modules $\mathcal{W}^s_{T^*X}$ on $T^*X$ is given by

$$\mathcal{W}^s_{T^*X} := R^1 a_! \mathcal{W}_{\mathbb{C}_s \times T^*X}.$$
The sheaf $\mathcal{W}_{T^*X}^s$

**Theorem**

(i) The sheaf $\mathcal{W}_{T^*X}^s$ is naturally endowed with a structure of a filtered $k$-algebra and $\text{gr } \mathcal{W}_{T^*X}^s \simeq R^1 a! \mathcal{O}_{\mathbb{C}^s \times T^*X}[[\hbar, \hbar^{-1}]]$.

(ii) Consider two complex manifolds $X$ and $Y$, two open subsets $U_X \subset T^*X$ and $U_Y \subset T^*Y$ and a symplectic isomorphism $\psi : U_X \sim \rightarrow U_Y$. Then, locally, $\psi$ may be quantized as an isomorphism of filtered $k$-algebras $\Psi : \mathcal{W}_{T^*X}^s \sim \rightarrow \mathcal{W}_{T^*Y}^s$ such that the isomorphism induced on the graded algebras coincides with the isomorphism $R^1 a! \mathcal{O}_{\mathbb{C}^s \times T^*X}[[\hbar, \hbar^{-1}]] \sim \rightarrow R^1 a! \mathcal{O}_{\mathbb{C}^s \times T^*Y}[[\hbar, \hbar^{-1}]]$ induced by $\psi$.

(iii) Assume $X$ is affine. There is an isomorphism of filtered sheaves of $k$-modules (not of algebras), called the “total symbol” morphism:

$$\sigma_{\text{tot}} : \mathcal{W}_{T^*X}^s \sim \rightarrow \mathcal{O}_{T^*X}^{s,\hbar}. \quad (3)$$

The total symbol of a product is given by the Leibniz formula with a convolution product in the $s$ variable.
The sheaf $\mathcal{W}_X^{s_{T^*X}}$

- Assume $X$ affine. For each Stein open subset $W$ of $T^*X$ and each relatively compact open subset $U \subseteq W$, sections of order $m$ are described by:

$$\sigma_{\text{tot}}(P)(s, x; \xi, \hbar) = \sum_{-\infty < j \leq m} p_j(s, x; \xi) \hbar^{-j}$$

- $p_j \in \Gamma(((\mathbb{C}_s \setminus K_0) \times U, \mathcal{O}_{\mathbb{C}_s \times T^*X})$

- $p_j$ satisfies canonical estimates on $K \subset (\mathbb{C}_s \setminus K_0) \times U$.

- The symbolic calculus is given by:

$$\sigma_{\text{tot}}(P \circ Q) = \sum_{\alpha \in \mathbb{N}^n} \frac{\hbar^{\alpha}}{\alpha!} \partial^\alpha_\xi \sigma_{\text{tot}}(P) \ast_c \partial^\alpha_x \sigma_{\text{tot}}(Q).$$
The sheaf $\mathcal{W}^t_{T^*X}$

- $\mathcal{W}^t_{T^*X}$ is a filtered sheaf of $\mathbf{k}$-algebras (algebra of exponentials).
- Locally, a section $P$ of order $m$ of $\mathcal{W}^t_{T^*X}$ on a Stein open subset $V$ of $T^*X$ and an open subset $U \subseteq V$, $\sigma_{\text{tot}}(P)$ is written as a series: $\sigma_{\text{tot}}(P)(t, x; \xi, \hbar) = \sum_{j \in \mathbb{Z}} p_j(t, x; \xi) \hbar^{-j}$
- $\forall K \subset U \subset T^*X$, $\exists \eta > 0$:
  - $p_j(t, x; \xi)$ is holomorphic around $\{|t| \leq \eta\} \times K$
  - $\exists C, \varepsilon > 0$, $\sup_{(x; \xi) \in K, |t| \leq \eta} |p_j(t, x; \xi)| \leq C \varepsilon^{-j}(-j)!$ for all $j < 0$.
  - $\exists M, R > 0$, $\sup_{(x; \xi) \in K} |p_j(t, x; \xi)| \leq M \frac{R^{j-m}}{(j-m)!} |t|^{j-m}$, $\forall |t| \leq \eta \ \forall j \geq m$.
- Symbolic calculus: usual Leibniz product.
- $\mathcal{W}^t_{T^*X}$ contains $\mathcal{W}^t_{T^*X}$ as a subalgebra.
Consider a section $P$ of $\mathcal{W}_{T^*X}(0)$ on an open subset $U$ of $T^*X$.

For each compact subset $K$ of $U$, there exists $R > 0$ such that the section $s - P$ of $\mathcal{W}^s_{T^*X}$ defined on $\mathbb{C}_s \times U$ is invertible on $(\mathbb{C}_s \setminus D(0, R)) \times K$, ($D(0, R)$ closed disc of radius $R$).

$\frac{1}{s - P}$ defines an element of $\Gamma(U; \mathcal{W}^s_{T^*X})$.

Expand $\frac{1}{s - P}$ as $\sum_{n\geq 0} \frac{P^n}{s^{n+1}}$ and apply Laplace transform.

Denote by $\exp(t\hbar^{-1}P)$ the image by $\mathcal{L}$ of $\frac{1}{s - P}$. 
Exponential elements

**Theorem**

For \( P \in \mathcal{W}_{T^*X}(0) \) (order 0), there is a section \( \exp(t\hbar^{-1}P) \in \mathcal{W}^{t}_{T^*X} \) such that, (when \( X \) is affine):

\[
\sigma_{tot}(\exp(t\hbar^{-1}P)) = \sum_{n \geq 0} \frac{(t\hbar^{-1}\sigma_{tot}(P))^{*n}}{n!},
\]

where the star-product \( f^{*n} \) means the product given by the Leibniz formula.