From uniform to partial hyperbolicity: outline of lectures

Amie Wilkinson

ESI, June 16, 18, 20, 2008

1 Introduction

1. **Partial hyperbolicity** $f\colon M \to M$ is *partially hyperbolic* if

   • $\exists$ a nontrivial Tf-invariant splitting $TM = E^s \oplus E^c \oplus E^u$, and

   • $\exists$ a Riemannian metric and continuous positive functions ν, $\hat{\nu}$, γ and $\hat{\gamma}$ with

   $$\nu, \hat{\nu} < 1 \quad \text{and} \quad \nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}$$

   such that, for any unit vector $v \in T_pM$,

   $$\begin{aligned} \|Tfv\| &< \nu(p), && \text{if } v \in E^s(p),\\ \gamma(p) < \|Tfv\| &< \hat{\gamma}(p)^{-1}, && \text{if } v \in E^c(p),\\ \hat{\nu}(p)^{-1} &< \|Tfv\|, && \text{if } v \in E^u(p). \end{aligned}$$

   Examples: time-1 map of Anosov flow, skew products. Note partially hyperbolic with E^c trivial is Anosov.

2. **"Classical" results in Anosov theory**:

   (a) **Ergodicity.**
   Theorem [Anosov] *f C^2, Anosov, volume-preserving $\Rightarrow$ ergodic (Bernoulli).*

   (b) **"Livsic" theorems.** [Livsic, L-Sinai, Guillemin-Kahzdan, de la Llave-Marco-Moriyon...] *Cohomological equation:*

   $$\phi = \Phi \circ f - \Phi \tag{1}$$

Theorem *Let $f\colon M \to M$ be an Anosov diffeomorphism and let $\phi : M \to \mathbf{R}$ be Holder continuous.*

I. Existence of solutions. *If f is C^1 and transitive, then (1) has a continuous solution Φ if and only if $\int_\gamma \phi = 0$, for every periodic orbit of f. Every continuous solution is Holder continuous.*

II. Measurable rigidity. *Let f be C^2 and volume-preserving. If there exists a measurable solution Φ to (1), then there is a continuous solution $\hat{\Phi}$, with $\hat{\Phi} = \Phi$ a.e.*

III. Higher regularity. *Let $k \geq 1$, and suppose that f and ϕ are C^k. Then every continuous solution to (1) is C^r, for every $r < k$.*

3. **Counterparts in partially hyperbolic theory**:

   (a) **Ergodicity.**

   Theorem A [Burns-W] f C^2, *volume-preserving, center bunched. If f is (essentially) accessible, then f is ergodic (K-system).*

   (b) **Livsic theorems.**

   Katok-Kononenko (1996): Studied *relative cohomological equation:*

   $$\phi = \Phi \circ f - \Phi + c. \tag{2}$$

   Defined an obstruction to solving this equation: *Periodic cycles functional:*

   $$PCF\colon \{\text{closed } su\text{-loops}\} \times C^\alpha(M) \to \mathbf{R}.$$

   Theorem B [W] *Let $f\colon M \to M$ be partially hyperbolic and accessible, and let $\phi : M \to \mathbf{R}$ be Hölder continuous*

   I. Existence of solutions. *If f is C^1, then (2) has a continuous solution Φ if and only if $PCF_\gamma(\phi) = 0$, for every su-path γ. Every continuous solution is Hölder continuous.*

   II. Measurable rigidity. *Let f be C^2, center bunched, and volume-preserving. If there exists a measurable solution Φ to (1), then there is a continuous solution $\hat{\Phi}$, with $\hat{\Phi} = \Phi$ a.e.*

   III. Higher regularity. *Let $k \geq 2$ be an integer. Suppose that the skew product $f_\phi(x,t) = (f(x), t + \phi(x))$ is C^k and r-bunched, for some $r < k$. Then every continuous solution to (1) is C^r.*

4. **Explanation of hypotheses**
    (a) *Accessibility:* $\exists$ foliations $\mathcal{W}^u$, $\mathcal{W}^s$. Definition of su-path
    (b) *PCF functional:* $x' \in \mathcal{W}^s(x)$:

    $$PCF_{x,x'}\phi = \sum_{i=0}^{\infty} \phi(f^i(x)) - \phi(f^i(x'))$$

    $x' \in \mathcal{W}^u(x)$:

    $$PCF_{x,x'}\phi = \sum_{i=1}^{\infty} \phi(f^{-i}(x)) - \phi(f^{-i}(x')).$$

    Definition then extends to su-paths.
    (c) *Center bunching:* $\nu < \gamma\hat{\gamma}$ and $\hat{\nu} < \gamma\hat{\gamma}$.
    (d) *r-bunching:*

    $$\nu < \gamma^r, \qquad \hat{\nu} < \hat{\gamma}^r \tag{3}$$

    $$\nu < \gamma\hat{\gamma}^r, \quad \text{and} \quad \hat{\nu} < \hat{\gamma}\gamma^r. \tag{4}$$

    Remark Anosov diffeos and flows are ∞-bunched.

5. **Outline of talks/main concepts** Will place in a framework with broader application.
    (a) Notation and basic concepts
    (b) Saturation and essential saturation
    (c) Juliennes and density
    (d) Homogeneity and regularity

2 Notation and basic concepts

1. **Notation:** f will always denote a partially hyperbolic diffeo. m is always volume. m_τ is induced volume on τ. x_n for $f^n(x)$. Write $A = B$ (μ-a.e.) if $\mu(A \,\Delta\, B) = 0$.

2. f $C^2 \Rightarrow \mathcal{W}^u$ and $\mathcal{W}^s$ are *absolutely continuous with bounded jacobians:* for $* \in \{s, u\}$, $\exists C \geq 1$ such that $\forall$ uniformly-chosen foliation box B, transversal τ to $\mathcal{W}^*$ and $A \subset B$:

$$C^{-1}m(A) \leq \int_\tau m_{\mathcal{W}^*}(A \cap \mathcal{W}^*(x))\, dm_\tau(x) \leq Cm(A).$$

3. *Dynamical coherence:* $E^u \oplus E^c$ and $E^s \oplus E^c$ are integrable. Assume sometimes for simplicity, but **not necessary for any results stated here.**

4. If f is center bunched and dynamically coherent, then holonomy between center leaves is C^1. r-bunched $\Rightarrow$ holonomy C^r.

3 Saturation and essential saturation.

1. **Saturated sets**
   (a) $\mathcal{F}$ foliation of M. $A \subseteq M$ is *$\mathcal{F}$-saturated* if $x \in A \Rightarrow \mathcal{F}(x) \subset A$
   (b) μ=measure on M. $A \subseteq M$ is *essentially $\mathcal{F}$-saturated (w.r.t. μ)* if $\exists \hat{A}$, $\mathcal{F}$-saturated, with $A = \hat{A}$. $\hat{A}$ is called an *$\mathcal{F}$-saturate* of A.
   (c) Let f be ph. A is
      i. *bi-saturated* if it is both $\mathcal{W}^s$ and $\mathcal{W}^u$ saturated
      ii. *bi-essentially saturated* if it is both essentially $\mathcal{W}^s$ and essentially $\mathcal{W}^u$ saturated
      iii. *essentially bi-saturated* if $\exists A^{su}$ bisaturated, with $A^{su} = A$ (μ-a.e.).
   (d) **Saturation and ergodicity** (the Hopf argument)

   Proposition 1. *f preserving μ is ergodic $\iff$ every f-invariant, bi-essentially saturated set w.r.t. μ has null or conull measure*

   f is a K system with respect to μ if every bi-essentially saturated w.r.t. μ has null or conull volume.

   Proof. Projection onto invariant functions conincides with Birkhoff averaging $\mathcal{B}^+$ and $\mathcal{B}^-$. Image of continuous functions is dense. The image of $C^0(M)$ under $\mathcal{B}^+$ is contained in functions constant along $\mathcal{W}^s$ and under $\mathcal{B}^1$ is contained in functions constant along $\mathcal{W}^-$. If f not ergodic, $\exists$ invariant essentially bisaturated set.

   Brin and Pesin proved that the Pinsker algebra of f with respect to μ is contained in the σ-algebra of $\mathcal{W}^s$-saturated sets. Applying inverses, it is also contained in the σ-algebra of $\mathcal{W}^u$-saturated sets. Hence the Pinsker algebra is constained in the σ-algebra of bi-essentially saturated sets; if this algebra is trivial, then f is a K-system with respect to μ.

*************** End of Lecture 1 **********************

2. **Saturated sections**

   (a) Setup $X \hookrightarrow B \to M$ is an *admissible bundle* if X is a Polish space and $\mathcal{W}^u$, $\mathcal{W}^s$ lift to foliations $\widehat{\mathcal{W}}^u$, $\widehat{\mathcal{W}}^s$ of B. (that is the charts can be chosen of the form $R^m \times R^{d-m} \times X$ compatible with $R^m \times R^{d-m}$ charts for $\mathcal{W}^s$, etc).

   (b) a section $\sigma : M \to B$ is *$*$-saturated* if $\sigma(M)$ is $\widehat{\mathcal{W}}^*$ saturated, *bi-saturated* if it is both s and u-saturated, *essentially bi saturated (w.r.t. μ on M)* if $\exists\, \hat{\sigma}$ bisat with $\hat{\sigma} = \sigma$, μ=a.e., and *bi-essentially saturated* if $\exists\, \sigma^s$ s-saturated, and σ^u, u-saturated, s.t. $\sigma = \sigma^s = \sigma^u$, μ-a.e.

3. **Examples**

   (a) **Characteristic functions.** Let $B = M \times \{0, 1\}$, trivial lift of foliations. For $A \subseteq M$, and $x \in M$, define $\sigma_A(x) = (x, 1_A(x))$. Then saturation properties of A correspond to saturation properties of σ_A.

   (b) **Abelian cocycles** Let $B = M \times R$.

      i. For $\phi : M \to R$, define f_ϕ on $M \times R$ by

      $$f_\phi(x, t) = (f(x), t + \phi(x)).$$

      ii. **Fact:** ϕ Holder $\Rightarrow \exists\, \widehat{\mathcal{W}}^u$, $\widehat{\mathcal{W}}^s$, covering $\mathcal{W}^u$, $\mathcal{W}^s$ s.t.

         A. f_ϕ-invariant

         B. $(x', t') \in \widehat{\mathcal{W}}^s(x, t) \iff x' \in \mathcal{W}^s(x)$ and

         $$\liminf_{n\to\infty} d(f^n_\phi(x, t), f^n_\phi(x', t')) = 0.$$

      iii. **Saturation and the relative cohomological equation:**

      $$\phi = \Phi \circ f - \Phi + c. \tag{5}$$

      Proposition 2. Let $\sigma_\Phi(x) = (x, \Phi(x))$.

         A. Φ *continuous solution to (5) $\Rightarrow$ σ_Φ bi-saturated.*

         B. *If f preserves m and is ergodic, then Φ measurable solution to (5) (m-a.e.) $\Rightarrow$ σ_Φ bi-essentially saturated (w.r.t m).*

Proof. A. Suppose that Φ is a continuous solution to (5). Then (5) implies that for all $x \in M$ and all n, we have:

$$f_\phi^n(x, \Phi(x)) = (f^n(x), \Phi(f^n(x)) + cn).$$

Let $x' \in \mathcal{W}^s(x)$. Then

$$\liminf_{n\to\infty} d(f_\phi^n(x,t), f_\phi^n(x',t')) =$$

$$\lim_{n\to\infty} d((f^n(x), \Phi(f^n(x))), (f^n(x'), \Phi(f^n(x')))) = 0,$$

and so x, x' lie on the same $\widehat{\mathcal{W}}^s$ leaf. This implies that σ_Φ is s-saturated. Similarly σ_Φ is u-saturated, and hence bisaturated.

B. Let Φ be a measurable solution to (5). We may assume that (5) holds on an f-invariant set of full volume; for points in this set, we have

$$f_\phi^n(x, \Phi(x)) = (f^n(x), \Phi(f^n(x)) + cn),$$

for all n.

Choose a compact set C of small covolume on which Φ is uniformly continuous. Ergodicity and absolute continuity of $\mathcal{W}^s$ implies that almost every pair of points on the same $\mathcal{W}^s$ leaf will visit C simultaneously for a positive density set of times. For such a pair of points x, x' we have

$$\liminf_{n\to\infty} d(f_\phi^n(x,t), f_\phi^n(x',t')) =$$

$$\liminf_{n\to\infty} d((f^n(x), \Phi(f^n(x))), (f^n(x'), \Phi(f^n(x')))) = 0,$$

and so x, x' lie on the same $\widehat{\mathcal{W}}^s$ leaf. This implies that σ_Φ is ess s-saturated. Similarly σ_Φ is ess u-saturated, and hence bi-essentially saturated.

(c) **Diff-valued cocycles and measure-valued sections (time-permitting):** Let N be a Riem mfld, $\varphi : M \to \mathrm{Diff}(N)$. Define $f_\varphi(x,p) = (f(x), \varphi_x(p))$.

   i. **Fact** φ β-Holder and *dominated* (i.e. $\nu(x)^\beta < m(D\varphi_x)$ and $\hat{\nu}(x)^\beta < m(D\varphi_x^{-1})$) $\Rightarrow$ $M \times N$ is admissible (and lifted foliations are f_φ-invariant)

   ii. Let $\mathcal{P}(N)$ = probability measures on N, weak* topology. Map $f_{\varphi,*}$ on $B = M \times \mathcal{P}(N)$. Also an admissible bundle.
   iii. **Theorem [Avila-Viana]** Let μ be f-invariant, and let $\sigma : M \to B$ be an $f_{\varphi,*}$-invariant section. Suppose that the Lyapunov exponents of f_φ with respect to σ vanish μ-a.e. Then σ is bi-essentially saturated (w.r.t. μ). (**Remark** Ledrappier)

4. **Accessibility and bi-saturation**

   (a) **Sets** Accessibility $\iff$ every bisaturated set is trivial ($= M$ or $\emptyset$). Essential accessibility $\iff$ every essentially bisaturated set (w.r.t. m) is trivial (has m-measure 0 or 1).

   (b) **Proposition 3** *Suppose f accessible. Then σ bisaturated $\Rightarrow$ σ continuous.*

   Proof. Baire Category Thm plus accessibility $\Rightarrow \forall x$, $\exists$ nbd U and map $\Gamma\colon U \times [0,1] \to M$ such that $\Gamma_y(0) = x$; $\Gamma_y(1) = y$, and

$$\lim_{y \to x} \Gamma_y = \Gamma_x.$$

   This implies every x is a point of continuity of bisat σ: consider lift $\hat{\Gamma}$ and use saturation of σ.

   (c) **Proposition 4** [Criterion for existence of bisaturated section] *Let $B \to M$ be admissible. Let $z \in B$ and let $x = \pi(z)$. There exists a bisaturated section $\sigma\colon B \to M$ with $\sigma(x) = z$ $\iff$ every closed su-loop in M through x lifts to a closed $\hat{su}$-loop in B through z.*

   Proof. Define σ by lifting paths. The condition implies that σ is well-defined.

   Relation with PCF functional in abelian cocycle context: $PCF_\gamma(\phi) = 0 \iff$ *every* lift of γ to B is closed.

   Corollary: Theorem B, part I. *f C^1 and accessible. $\exists$ continuous solution $\iff$ $PCF_\gamma(\phi) = 0$, for every closed su-loop γ.*

   Proof. By Proposition 4. $PCF_\gamma(\phi) = 0$, for every closed su-loop $\gamma \Rightarrow \exists$ bi-saturated section $\sigma : M \to M \times R$. Proposition 3 plus accessibility implies this section is continuous, so $\exists$ a continuous solution.

   On the other hand, if there exists a continuous solution Φ to (5), then Proposition 2 implies that σ_Φ is bisaturated. Proposition 4 $\Rightarrow PCF_\gamma(\phi) = 0$, $\forall$ closed su-loop γ.

(d) **Theorem C** *Let f be C^2 and center-bunched.*

  i [Burns-W] $A \subseteq M$ *bi-essentially saturated (w.r.t. m)* $\Rightarrow A$ *essentially bisaturated (w.r.t. m)*

  ii. [Avila-Santamaria-Viana] $B \to M$ *admissible.* $\sigma : B \to M$ *bi-essentially saturated section (w.r.t m)* $\Rightarrow \sigma$ *essentially bi-saturated (w.r.t. m)*

Proof. Next lecture.

(e) **Corollaries and consequences**

  i. **Theorem A:** f C^2 v.p. CB essentially accessible $\Rightarrow$ ergodic.
  Proof. By Proposition 1, it suffices to show that every bi-essentially saturated set A has null or conull volume. Let A be bi-ess sat. Theorem C [i] $\Rightarrow A$ is ess bi-sat. Then f is essentially accessible, $m(A)$ is 0 or 1.

  ii. **Part II of Theorem B:** f C^2 v.p. CB accessible. Φ measurable solution to (5) $\Rightarrow \exists\ \hat{\Phi}$ continuous solution.
  Proof. Note that f is ergodic, by Theorem A. Let Φ be a measurable solution to (5). Proposition 2 implies that Φ is bi-essentially saturated. Theorem C implies that σ_Φ is ess bi-sat, so there exists $\hat{\Phi} = \Phi$ a.e. with $\sigma_{\hat{\Phi}}$ bi-saturated. Proposition 3 plus accessibility imply $\sigma_{\hat{\Phi}}$ (and so $\hat{\Phi}$) is continuous.

  iii. **Theorem [A-S-V]** (time-permitting.) *f v.p. CB accessible. The generic bunched linear (GL_n or SL_n) cocycle over f has distinct extremal exponents.*
  Sketch of proof. Let $A : M \to GL(n, V)$ be a linear cocycle. Consider the projective cocycle $\varphi_A : M \to \mathrm{Diff}(P^n(V))$. Bunching of A implies domination of φ_A implies admissibility of $M \times N$, where $N = P^n(V)$.
  Suppose that the extremal exponents of A coincide. This implies the lyapunov exponents of φ_A vanish. Then [Avila-Viana] implies bi-essential saturation of any $f_{\varphi_A,*}$-invariant section of $M \times \mathcal{P}(N)$. Theorem C implies any such section is essentially bisaturated. Invariant sections exist by standard (Bogolubev-Krylov) averaging arguments. Hence if the extremal exponents of A coicide, there exists a (continuous) section of $M \times \mathcal{P}(N)$, invariant under lifted holonomy maps. But by examining enough *su*-loops one sees that for generic A, there can be no holonomy-invariant measures, so no bisaturated sections. Hence generically extremal exponents are different.

**********************End of Lecture 2**********************

4 Juliennes and density

Recall Theorem C. The proof of (ii) follows closely the proof of (i), so we prove (i).

1. **Refined statement of theorem C (i):** *Let f be CB. A bi-ess sat $\Rightarrow DP(A)$ bisat, where $DP(A) =$ set of Lebesgue density points of A.*
2. **Simplifying assumptions:** dynamical coherence, $\nu, \hat{\nu}, \gamma, \hat{\gamma}$ constant.
3. **Choice of σ, τ:** $\sigma < \min\{1, \hat{\gamma}\}$ and $\nu < \tau < \sigma\gamma$.
4. **Definition of juliennes:** $J_n^{cu}(x) = \bigcup_{y \in \mathcal{W}^c(x,\sigma_n)} f^{-n}(\mathcal{W}^u(y, \tau^n))$.
5. **Key properties of juliennes**
   (a) x is a d.p. of a bi-essentially saturated set $A \iff x$ is a julienne density point of any $\mathcal{W}^s$-saturate A^s
   (b) x is a julienne d.p. of a saturated set A^s, $y \in \mathcal{W}^s(x) \Rightarrow y$ is a julienne d.p. of A^s
6. **Two tools:** equivalence between indexed sequences of sets: *internestedness* and *fibered equivalence.*
7. **Equivalence of sequences of sets:** Picture. Uses absolute continuity with bounded jacobians.
8. **Adapting to get proof of Theorem C** Let $\Phi \colon M \to X$ be measurable. Say $x \in M$ is a point of *measurable continuity* if $\exists z \in X$ such that x is a Lebesgue density point of $\Phi^{-1}(U)$, for every neighborhood U of x. z is called the *density value* of Φ at x. The set of points of measurable continuity has full measure. On this set define $\hat{\Phi}(x)$ to be the density value of Φ at x. Then $\hat{\Phi} = \Phi$ m-a.e.

   The main result is that if σ_Φ is bi-essentially saturated, then $\sigma_{\hat{\Phi}}$ is bi-saturated. The proof is to show that points of measurable continuity are bi-saturated, as are the density values (by the lifted foliations).

5 Homogeneity and regularity

1. **Journé's Theorem** If $\Phi : M \to R$ is C^r along leaves of two transverse foliations with uniformly C^r leaves, then Φ is C^r.

2. **The argument when f is Anosov:** Let Φ be a continuous solution of (5). Proposition 2A $\Rightarrow$ σ_Φ is bisaturated. The graph of Φ over $\mathcal{W}^*$ is the leaf $\widehat{\mathcal{W}}^*$, which is C^r. Hence Φ is C^r restricted to $\mathcal{W}^*$ leaves. Journé $\Rightarrow$ Φ is C^r.

3. **Regularity in the center direction**

4. **Theorem D:** *If X is a C^r homogeneous submanifold of R^n, then X is a C^r submanifold.*

   Remark: cf. [Bochner-Montgomery], [Repovš-Skopenkov-Sčepin].

   Idea of proof. Picture. Show Lipschitz first. $\Rightarrow$ diff'ble a.e. Homogeneity $\Rightarrow$ C^1. Inductive argument on jets gives C^r.

5. **Proof of C^r regularity of solutions to cohomological equation (Theorem B, part III**).

   Assume dynamical coherence for simplicity. Let Φ be a solution of (5). The restriction of Φ to $\mathcal{W}^s$ leaves and $\mathcal{W}^u$ leaves is C^r, by Anosov argument above. The restriction of Φ to $\mathcal{W}^c$ leaves is C^r-homogeneous, by accessibility and r-bunching. Theorem D $\Rightarrow$ the restriction of Φ to $\mathcal{W}^c$ leaves is C^r. Apply Journé twice, first to $\mathcal{W}^s$ and $\mathcal{W}^c$ leaves, to obtain uniformly C^r along $\mathcal{W}^{cs}$ leaves, then to $\mathcal{W}^{cs}$ leaves and $\mathcal{W}^u$ leaves, to obtain Φ is C^r.