Another Veech triangle

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April 6, 2007

The goal of this paper is to demonstrate that the triangle Δ with angles $(\frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12})$ has Veech's lattice property. See figure 1 for a picture of this triangle and the translation surface, X_Δ, obtained by unfolding the triangle. Beginning with work of Veech, the lattice property for polygons has been shown to have profound implications for the dynamics of billiard trajectories. See [Vee89], for instance.

This example fits into an infinite class of surfaces with the lattice propery of genera 3 and 4 discovered by McMullen [McM].

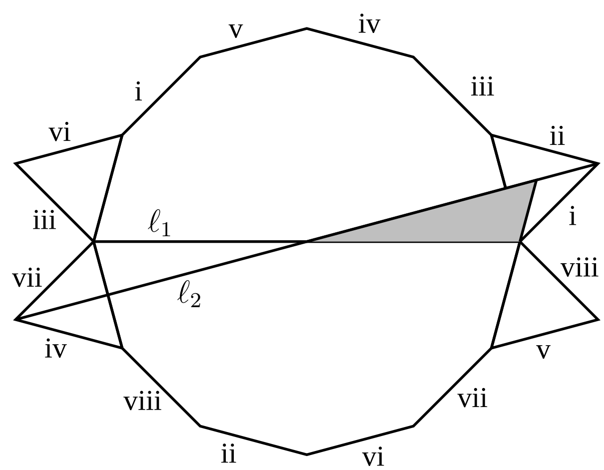

Figure 1: The triangle $\Delta = (\frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12})$ together with the corresponding translation surface, X_Δ. Roman numerals indicate edge identifications.

We also explicitly describe the affine automorphism group of X_Δ, which is a subgroup of

$$\widehat{SL}(2,\mathbb{R}) = \{M \in GL(2,\mathbb{R}) | Det(M) = \pm 1\} \tag{1}$$

Theorem 1. *The surface X_Δ has the lattice property. A fundamental domain for the action of the affine automorphism group $\Gamma(X_\Delta) \subset \widehat{SL}(2,\mathbb{R})$ on the hyperbolic plane is shown in figure 2. $\Gamma(X_\Delta)$ is generated by reflections in the sides of this polygon together with $-I$.*

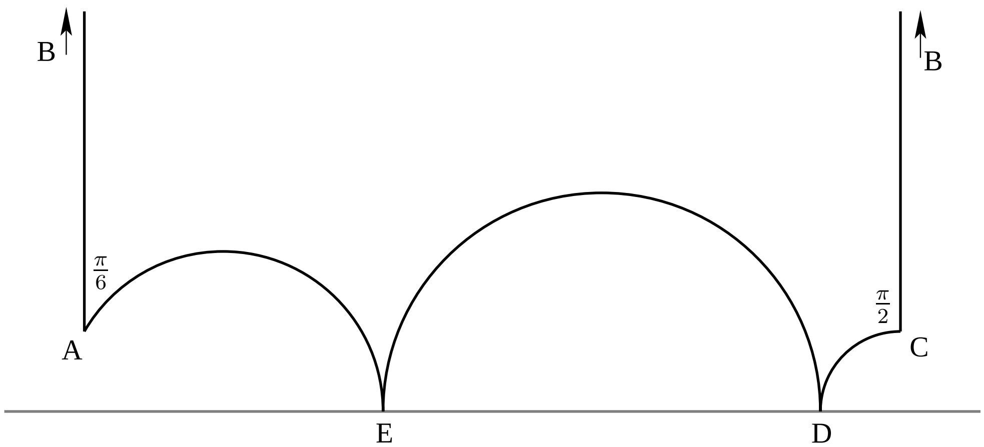

Figure 2: The fundamental domain for the action of $\Gamma(MT_\Delta)$ on the upper half-plane is the polygon pictured. The polygon is the hyperbolic convex hull of its vertices: $A = i$, $B = \infty$, $C = 5 + 3\sqrt{3} + i$, $D = 4 + 3\sqrt{3}$, and $E = 2 + \sqrt{3}$.

This adds an additional triangle to the list of known triangles with the lattice property. This list follows.

1. The acute isosceles triangles with angles $(\frac{(n-1)\pi}{2n}, \frac{(n-1)\pi}{2n}, \frac{\pi}{n})$ for $n \geq 3$. (due to Veech [Vee89]).

2. The acute triangles $(\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12})$, $(\frac{\pi}{5}, \frac{\pi}{3}, \frac{7\pi}{15})$, $(\frac{2\pi}{9}, \frac{\pi}{3}, \frac{4\pi}{9})$. (due to Veech [Vee89], Vorobets [Vor96], and Kenyon and Smillie [KS00] respectively).

3. The right triangles with angles $(\frac{\pi}{n}, \frac{(n-2)\pi}{2n}, \frac{\pi}{2})$ for $n \geq 4$. (due to Veech [Vee89]).

4. The obtuse isosceles triangles with angles $(\frac{\pi}{n}, \frac{\pi}{n}, \frac{(n-2)\pi}{n})$ with $n \geq 5$. (due to Veech [Vee89]).

5. The obtuse triangles with angles $(\frac{\pi}{2n}, \frac{\pi}{n}, \frac{(2n-3)\pi}{2n})$ with $n \geq 4$. (due to Ward [War98]).

The work of Kenyon and Smillie [KS00] together with that of Puchta [Puc01] has shown that this list is complete in the case of acute, right, and isosceles triangles. Kenyon and Smillie provided a simple criterion which can be used to decide that a given triangle does not have the lattice property. This criterion requires knowledge of the existence of a periodic billiard trajectory. Kenyon and Smillie used the Fagnano curve, which is a periodic billiard path in every acute triangle, to rule out all but a short list of acute triangles with anlges that are rational multiples of π with denominator less than 10,001. Puchta later eliminated the remaining acute triangles.

The triangle Δ with angles $(\frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12})$ was found using the methods of Kenyon and Smillie. Rich Schwartz and the author have written a computer program called McBilliards[1] which is capable of finding periodic billiard trajectories in triangles. This program together with Kenyon and Smillie's criterion enabled the author to (non-rigorously) search for triangles which might satisfy the lattice property. The triangle Δ was the only new triangle with angles of small denominator which seemed to pass this test.

Open Question 2. *Is the list of obtuse triangles with the lattice property now complete?*

I would like to thank Barak Weiss for the idea of searching for obtuse Veech triangles, and Curt McMullen for his encouragement to write this result down. I would also like to thank Rich Schwartz for his collaboration on McBilliards and my graduate advisor Yair Minsky.

1 Background

In this section, we briefly define a translation surface, its affine automorphism group, and the lattice property. For more details see [MT02].

A *translation surface* is a closed oriented surface X together with a discrete set $\Sigma \subset X$ and an atlas of charts from $X \smallsetminus \Sigma$ to the plane so that the transition functions are translations. The subset Σ is a known as the *singular set*. The *atlas of charts* is a covering of $X \smallsetminus \Sigma$ by open sets U_i together with local homeomorphisms $\phi_i : U_i \to \mathbb{R}^2$. The *transition functions* are the maps $\phi_i \circ \phi_j^{-1}$ restricted to $\phi_j(U_i \cap U_j)$. The translation surface inherits the pull back metric from the plane and also the notion of direction. Small open sets of $X \smallsetminus \Sigma$ are thus isometric to the plane and the points of Σ are cone points that have cone angles which are integer multiples of 2π. The relevant example of a translation surface is shown in figure 1.

We will let $\widehat{SL}(2,\mathbb{R})$ be the subgroup of affine transformations of the plane that preserve area and fix the origin. See equation 1. An element $A \in \widehat{SL}(2,\mathbb{R})$ acts affinely on the plane. Given X we can form a new translation surface $A(X)$ by post composing the charts of X with A. The transition functions of $A(X)$ are translations, since they are just the transition functions of X conjugated by A. Thus, $A(X)$ is another translation surface.

The *affine automorphism group*, $\Gamma(X) \subset \widehat{SL}(2,\mathbb{R})$, of X is the set of elements $A \in \widehat{SL}(2,\mathbb{R})$ so that there is a direction preserving isometry $\phi : X \to A(X)$. (Direction preserving is important, otherwise rotations would automatically be in $\Gamma(X)$.) A translation surface X is said to have the *lattice property* if $\Gamma(X) \subset \widehat{SL}(2,\mathbb{R})$ is a lattice.

Veech discovered a relevant and powerful lemma about parabolics in the affine automorphism group in terms of cylinders of the surface. See [Vee89] and

[1] McBilliards is freely available from `http://mcbilliards.sourceforge.net/`.

[MT02]. The *modulus* of a cylinder is the height of the cylinder divided by its circumference.

Lemma 3 (Veech). *There is a parabolic in the group $\Gamma(X)$ fixing the direction θ if and only if there is a decomposition of the surface into cylinders in the direction θ whose moduli are commensurable (rational multiples of one another).*

Remark 4. *Suppose θ is the horizontal direction and α is the greatest common divisor of the moduli of the cylinders in the horizontal decomposition given by the lemma. The greatest common divisor of a set of commensurable numbers $\{m_1, \ldots, m_n\}$ is the largest number α so that $\frac{m_i}{\alpha} \in \mathbb{Z}$ for all i. The generating parabolic fixing θ is given by*

$$\begin{pmatrix} 1 & \frac{1}{\alpha} \\ 0 & 1 \end{pmatrix}$$

To aid in visualizing $\Gamma(X)$, it is worth considering the action of $\widehat{SL}(2,\mathbb{R})$ on the upper half plane. The group $\widehat{SL}(2,\mathbb{R})$ of equation 1 acts on the upper half-plane by hyperbolic isometries in the standard way. The upper half-plane is a subset of the Riemann sphere, $\hat{\mathbb{C}} = \mathbb{C}^2/(\mathbb{C} \smallsetminus \{0\})$. The upper half plane is the equivalence classes of elements $(z,1) \in \mathbb{C}^2$ where z has positive imaginary part. An element of $\widehat{SL}(2,\mathbb{R})$ acts on $\hat{\mathbb{C}}$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{cases} \begin{pmatrix} az+bw \\ cz+dw \end{pmatrix} & \text{if } ad-bc=1 \\ \begin{pmatrix} a\overline{z}+b\overline{w} \\ c\overline{z}+d\overline{w} \end{pmatrix} & \text{if } ad-bc=-1 \end{cases} \tag{2}$$

Note that this action is not faithful because $-I$ acts trivially.

2 The Proof

We break up the proof of the theorem into two lemmas. In the first we prove that the elements we list are in $\Gamma(X_\Delta)$. Then we will show that this list generates all of $\Gamma(X_\Delta)$.

Lemma 5. *Each of the reflections in the side of the polygon of figure 2 is in $\Gamma(X_\Delta)$. $-I$ is also in $\Gamma(X_\Delta)$.*

Proof: The surface X_Δ has several Euclidean automorphisms. The element $-I$ acts on the plane by a Euclidean rotation by π. Thus, it is clear $-I \in \Gamma(X_\Delta)$. The Euclidean automorphism group is generated by reflections in lines ℓ_1 and ℓ_2 of figure 1. This gives two of our generators:

$$R_{\overline{AB}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad R_{\overline{AE}} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \tag{3}$$

Now we will find a parabolic automorphism of X_Δ fixing the point B. There is a decomposition of X_Δ by saddle connections parallel to ℓ_1 of figure 1. This decomposition is depicted in figure 3, and cuts the surface into 4 cylinders. It can be verified that these cylinders come in pairs with two possible moduli:

$$\frac{1}{5+3\sqrt{3}} \textit{ and } \frac{1}{10+6\sqrt{3}} \tag{4}$$

In particular, note the first modulus is twice the second. Thus, there is a parabolic element of the automorphism group which fixes the horizontal direction and acts as a single Dehn twist on the pair of cylinders with modulus $\frac{1}{10+6\sqrt{3}}$ and a double Dehn twist on the pair of cylinders with modulus $\frac{1}{5+3\sqrt{3}}$. This parabolic is:

$$P_B = \begin{pmatrix} 1 & 10+6\sqrt{3} \\ 0 & 1 \end{pmatrix} \tag{5}$$

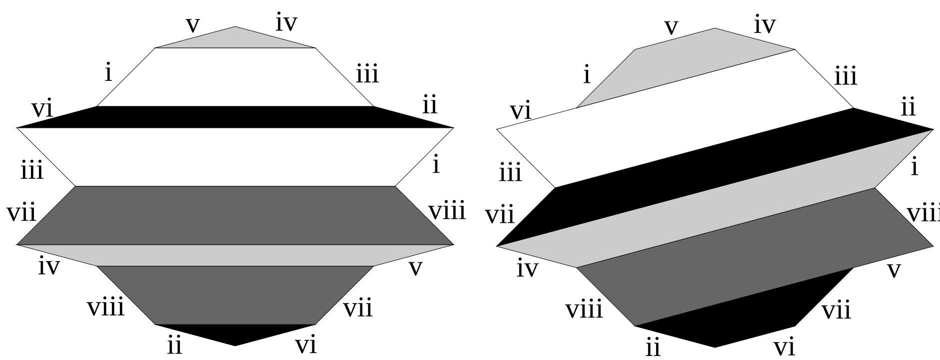

Figure 3: Decompositions into cylinders using saddle connections parallel to lines ℓ_1 and ℓ_2 of figure 1.

Both the reflection $R_{\overline{AB}}$ and the parabolic P_B fix the point B, so their composition does as well. Their composition gives another reflection:

$$R_{\overline{BC}} = P_B \circ R_{\overline{AB}} = \begin{pmatrix} 1 & -10-6\sqrt{3} \\ 0 & -1 \end{pmatrix} \tag{6}$$

The same idea will work for generating a reflection preserving points E and D. We decompose X_Δ into 4 cylinders using segments parallel to ℓ_2 of figure 1. This decomposition is shown on the right in figure 3. The moduli of these cylinders again come in pairs:

$$\frac{3}{6+4\sqrt{3}} \textit{ and } \frac{1}{6+4\sqrt{3}} \tag{7}$$

This means that there is a parabolic inducing a single Dehn twist on the cylinders with moduli $\frac{1}{6+4\sqrt{3}}$ and a triple Dehn twist on the other cylinders. This parabolic is

$$P_E = \begin{pmatrix} \frac{1}{2}(-1-2\sqrt{3}) & \frac{1}{2}(12+7\sqrt{3}) \\ \frac{1}{2}(-\sqrt{3}) & \frac{1}{2}(5+2\sqrt{3}) \end{pmatrix} \tag{8}$$

Again we obtain $R_{\overline{DE}}$ by composition:

$$R_{\overline{DE}} = R_{\overline{AE}} \circ P_E = \begin{pmatrix} \frac{1}{2}(-3-\sqrt{3}) & \frac{1}{2}(13+7\sqrt{3}) \\ \frac{1}{2}(1-\sqrt{3}) & \frac{1}{2}(3+\sqrt{3}) \end{pmatrix} \tag{9}$$

We apply the same trick one last time. $R_{\overline{DE}}$ preserves two parallel families of lines on the surface, each corresponding to eigenvectors of the matrix. Of course, one is the family of lines parallel to ℓ_2. The second family has slope $\frac{1}{11}(-4+3\sqrt{3})$. We decompose the surface using saddle connections parallel to this direction (see figure 4). Again, these cut the surface into four cylinders whose moduli come in pairs. The moduli are

$$\frac{1}{29+17\sqrt{3}} \text{ and } \frac{1}{58+34\sqrt{3}} \tag{10}$$

Therefore, we get a parabolic fixing lines of slope $\frac{1}{11}(-4+3\sqrt{3})$:

$$P_D = \begin{pmatrix} \frac{1}{2}(-11-7\sqrt{3}) & \frac{1}{2}(115+67\sqrt{3}) \\ \frac{1}{2}(-1-\sqrt{3}) & \frac{1}{2}(15+7\sqrt{3}) \end{pmatrix} \tag{11}$$

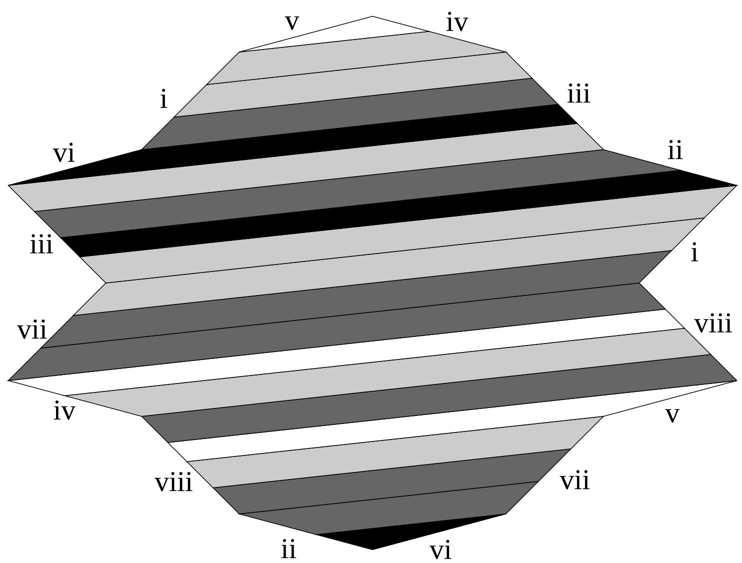

Figure 4: Saddle connections with slope $\frac{1}{11}(-4+3\sqrt{3})$ decompose the surface into these cylinders.

We compose the parabolic with the reflection $R_{\overline{DE}}$:

$$R_{\overline{CD}} = R_{\overline{DE}} \circ P_D = \begin{pmatrix} 5+3\sqrt{3} & -51-30\sqrt{3} \\ 1 & -5-3\sqrt{3} \end{pmatrix} \tag{12}$$

One way to check that lines $\overline{BC}$ and $\overline{CD}$ intersect at a right angle is to compute the trace of the product of reflections in the sides. If this trace is zero, then the sides meet at a right angle. We compute

$$Tr(R_{\overline{BC}} \circ R_{\overline{CD}}) = Tr \begin{pmatrix} -5-3\sqrt{3} & 53+30\sqrt{3} \\ -1 & 5+3\sqrt{3} \end{pmatrix} = 0 \tag{13}$$

$\diamondsuit$

Lemma 6. *The reflections in the side of the polygon of figure 2 together with* $-I$ *generate the Veech group* $\Gamma(MT_\Delta)$.

Proof: Let G be the group generated by $-I$ and the reflections in the sides of the polygon of figure 2, and let G^+ be the index 2 subgroup which preserves the orientation of $\mathbb{H}^2$.

Let $X_G = \mathbb{H}^2/G^+$, a sphere with 3 punctures and two cone singularities. One singularity has cone angle π and the other has cone angle $\pi/3$. We can compute the area of X_G using the Gauss-Bonnet formula. Recall the Gauss-Bonnet formula for hyperbolic surfaces with cone singularities tells us that for a surface S of genus g with p punctures and cone singularities of cone angle $\theta_1, \ldots, \theta_n$,

$$\mathit{area}(S) = 2\pi(2g+p-2) + \sum_{i=1}^{n}(2\pi - \theta_i) \tag{14}$$

We compute that $\mathit{area}(X_G) = \frac{14\pi}{3}$.

Now let V be the complete Veech group, V^+ be the orientation preserving subgroup, and $X_V = \mathbb{H}^2/V^+$. We wish to show $\Gamma(MT_\Delta) = G$. The previous lemma showed that G is a subgroup of V. Thus we have a covering map $\psi : X_G \to X_V$. Further we know that

$$[V^+ : G^+] = \mathit{area}(X_G)/\mathit{area}(X_V) \tag{15}$$

where $[V^+ : G^+]$ is the index of the subgroup G^+ inside V^+. In order to show that $V^+ = G^+$, it is sufficient to show $\mathit{area}(X_G)/\mathit{area}(X_V) < 2$.

We would like to use Gauss-Bonnet on X_V. First we will show that X_V also has 3 punctures. It is sufficient to show that none of the punctures of X_G can be identified by ψ. We will give affine invariants which distinguish the three decompositions into cylinders mentioned in the previous proof. The ratio of the moduli of the cylinders associated to vertex E of the polygons is 3, while the ratios of the moduli of cylinders associated to B and D are both 2 (see equations 7, 4, and 10). Thus E can not be identified with B or D. Another affine invariant is the ratio of the widths of the cylinders. We can compute that these ratios are

$$w_B = 1+\sqrt{3} \quad \text{and} \quad w_D = \tfrac{1+\sqrt{3}}{2} \tag{16}$$

Thus, the punctures coming from B and D cannot be identified by the covering ψ. This shows that X_V has 3 punctures.

We also need to show that X_V has at least one cone singularity. The image of a cone singularity in X_G must be a cone singularity in X_V. Further, the image of the cone singularity with cone angle $\pi/3$ must be a cone singularity with cone angle θ which is less than $\pi/3$. Gauss-Bonnet now tells us that

$$area(X_V) \geq 2\pi(3-2) + (2\pi - \theta) \geq 4\pi - \pi/3 = 11\pi/3 > \frac{1}{2} area(X_G) \tag{17}$$

Thus $area(X_G)/area(X_V) < 2$, so $[V^+ : G^+] = 1$ and $V^+ = G^+$.

Finally, because both V and G contain orientation reversing elements, we know $[V : G] = [V^+ : G^+]$. Thus $V = G$. $\diamondsuit$

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