ARNOLD DIFFUSION AND INSTABILITIES IN HAMILTONIAN DYNAMICS

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Abstract. We use variational method to study Arnold diffusion and instabilities in Hamiltonian systems. Our method is based on a generalization of Mather's theory on twist maps and their connecting orbits to a higher dimensional setting. Under some generic non-degeneracy conditions, we construct transition chains of arbitrary fixed length, crossing gaps of any size between invariant KAM (lower dimensional) tori. This solves the so-called gap problem. The most important feature of our result is that, instead of using transition tori for diffusion as in Arnold's construction [1], we also use cantori from Aubry-Mather theory in our mechanism for diffusion. We call this mechanism for diffusion Mather's mechanism. Other results, such as shadowing properties, symbolic dynamics, etc., can also be obtained by our method. Our non-degeneracy condition is a condition on the splitting of separatrix and in the so-called *a priori* unstable systems, this condition can be verified by the so-called Poincaré-Melnikov integrals.

1. Introduction

We consider the so-called *a priori* unstable systems:

$$H_\epsilon(\xi,\eta,I,\theta,t) = F(\xi,\eta) + G(I) + H^1(\xi,\eta,I,\theta,t,\epsilon),$$

where $\xi,\eta \in \mathbb{R}$, $I \in \mathbb{R}$ and $\theta \in \mathbb{T}$ and ϵ is a small parameter such that $H^1 = 0$ if $\epsilon = 0$. We make the following assumptions:

(1) Periodic dependence in t, i.e.,
$$H_\epsilon(\xi,\eta,I,\theta,t) \equiv H_\epsilon(\xi,\eta,I,\theta,t+1).$$
(2) The Hamiltonian flow on $\mathbb{R}^2$ generated by $F(\xi,\eta)$ has a hyperbolic rest-point P and there is homoclinic loop to the hyperbolic rest-point P, i.e., there exists a trajectory whose α-limit and ω-limit are both P.
(3) $G(I)$ is positive definite in I with super-linear growth. More precisely, $\partial^2 G(I)/\partial I_j^2 > 0$ everywhere and $G(I)/\|I\| \to \infty$ as $\|I\| \to \infty$.

We may state our main theorem, announced in Xia [8].

Theorem 1.1. *Fix constants A, B, $0 < A < B$. There exists an open set $\mathcal{U}$ of C^2 functions H^1 containing the zero function and an open and dense*

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subset $\mathcal{V} \subset U$ such that if $H^1 \in \mathcal{V}$, the above system is unstable: there exists a trajectory of the Hamiltonian flow which connects the region with $I < A$ to the region with $I > B$.

We remark that even though the theorem is stated in the form of a perturbation problem, our method is not limited to very small H^1. In fact, when H^1 is small, the open and dense set $\mathcal{V}$ stated in the theorem can be described by a weak non-degeneracy condition on the perturbation function H^1. This nondegenracy condition can be verified by a so-called Melnikov function. The precise formulation of this Melnikov condition will be given later.

The techniques introduced in this paper can be applied to a more general class of Hamiltonian systems in arbitrary dimensions. These applications include an extention of Mather's theorem on connecting orbits (Mather [6]). Detailed statements of results and proofs will appear in further publications.

Similar results in the context of geodesic flow was announced by Mather. The idea of the proof is different from what we present here in this paper.

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2. A review of the Aubry-Mather theory for monotone twist maps

In this section, we recall some of the basic results in Aubry-Mather theory for the monotone twist maps. These results will be used in later sections. The results are stated without proof. One can refer to Bangert citeBa88 for complete details.

Aubry-Mather theory studies the *Area-Preserving Monotone Twist Maps*. An area-preserving monotone twist map $f : S^1 \times [0,1] \to S^1 \times [0,1]$ is an area-preserving map with the following property: Let $\pi : S^1 \times [0,1] \to S^1$ be the projection of the annulus into its first component, and let l_a be the image of a vertical line $a \times [0,1], a \in S^1$, then $\pi : l_a \to \pi(l_a) \subset S^1$ is a diffeomorphism for every $a \in S^1$. We may extend the map on the annulus to a twist map on the whole cylinder $S^1 \times \mathbb{R}$.

Monotone twist maps can be described by a generating function $h : \mathbb{R}^2 \to \mathbb{R}$ in the following way: let $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f, then $\tilde{f}(x_0, y_0) = (x_1, y_1)$ is implicitly defined by the following two equations:

$$\begin{aligned} -D_1 h(x_0, x_1) &= y_0 \\ D_2 h(x_0, x_1) &= y_1 \end{aligned}$$

Here $D_i, i = 1, 2$ is the partial derivative with respect to the ith component, $h \in C^2$, $h(x_0 + 1, y_0 + 1) = h(x_0, y_0)$ and $D_2 D_1 h \leq -\delta < 0$.

Aubry-Mather theory studies the orbit structure of the twist maps by projecting orbits into their first components, or the configuration space. We consider the space $\mathbb{R}^{\mathbb{Z}} = \{x \mid x : \mathbb{Z} \to \mathbb{R}\}$ of bi-infinite sequences of real numbers with the product topology. An element $x \in \mathbb{R}^{\mathbb{Z}}$ is will also be denoted by $(x_i)_{i\in\mathbb{Z}}$ and will be called an orbit or a trajectory. Convergence of a sequence $x^n \in \mathbb{R}^{\mathbb{Z}}$ to $x \in \mathbb{R}^{\mathbb{Z}}$ means that $\lim_{n\to\infty} x_i^n = x_i$ for all $i \in \mathbb{Z}$.

Given a function $h : \mathbb{R}^2 \to \mathbb{R}$, we extend h to arbitrary finite segments $(x_j, \dots, x_k)$, $j < k$, of trajectories $x \in \mathbb{R}^{\mathbb{Z}}$ by

$$h(x_j, \dots, x_k) = \sum_{i=j}^{k-1} h(x_i, x_{i+1}).$$

We say a segment is *minimal* or *action-minimizing* with respect to h if

$$h(x_j, \dots, x_k) \le h(x_j^*, \dots, x_k^*)$$

for all $(x_j^*, \dots, x_k^*)$ with $x_j = x_j^*$ and $x_k = x_k^*$.

A point $x \in \mathbb{R}^{\mathbb{Z}}$ is *minimal* or *action-minimizing* with respect to h if every finite segment of x is minimal or action-minimizing with respect to h.

The set of all action-minimizing trajectories in $\mathbb{R}^{\mathbb{Z}}$ will be denoted by $\mathcal{M} = \mathcal{M}(h)$. Obviously, $\mathcal{M}$ is a closed subset in $\mathbb{R}^{\mathbb{Z}}$. The Aubry-Mather theory gives a fairly complete description of $\mathcal{M}$.

We assume that h is continuous and satisfies the following conditions $(H_1) - (H_4)$ (cf. Mather [3] and Bangert [2]):

(H_1) "Periodicity Condition": For all $(\xi, \eta) \in \mathbb{R}^2$, we have $h(\xi+1, \eta+1) = h(\xi, \eta)$;

(H_2) "Condition at Infinity": $\lim_{|\eta|\to\infty} h(\xi, \xi+\eta) = \infty$ uniformly in ξ;

(H_3) "Ordering Condition": If $\underline{\xi} < \bar{\xi}$, $\underline{\eta} < \bar{\eta}$, then

$$h(\underline{\xi}, \underline{\eta}) + h(\bar{\xi}, \bar{\eta}) < h(\underline{\xi}, \bar{\eta}) + h(\bar{\xi}, \underline{\eta})$$

(H_4) "Transversality Condition": If $(x_{-1}, x_0, x_1) \neq (x^*_{-1}, x^*_0, x^*_1)$ are both minimal and $x_0 = x_0^*$, then $(x_{-1} - x^*_{-1})(x_1 - x_1^*) < 0$.

The above four conditions constitute the basic requirements we have on h. It is a simple exercises to verify that if h is the generating function for the monotone twist map, where $h \in C^2$ and $D_2 D_1 h \le -\delta < 0$, then $(H_1) - (H_4)$ are all satisfied. To relate orbits in the configuration space and orbits for the twist map, we make the following definition:

Definition 2.1. *We say that $x \in \mathbb{R}^{\mathbb{Z}}$ is* stationary *if*

$$D_2 h(x_{i-1}, x_i) + D_1 h(x_i, x_{i+1}) = 0 \ \text{ for all } \ i \in \mathbb{Z}.$$

Obviously each $x \in \mathcal{M}(h)$ is stationary with respect to h. For any stationary orbit x, if we let $y_i = -D_1 h(x_i, x_{i+1}) = D_2 h(x_{i-1}, x_i)$, then $\tilde{f}(x_i, y_i) = (x_{i+1}, y_{i+1})$ for all $i \in \mathbb{Z}$, i.e., $\{(x_i, y_i)\}_{i\in\mathbb{Z}}$ forms a true orbit for the twist map in the phase space. More importantly, all the action-minimizing configurations correspond to true orbits in the phase space.

For any $x \in \mathbb{R}^{\mathbb{Z}}$, the *Aubry graph* of x is a graph in $\mathbb{R}^2$, where we connect two points (i, x_i) and $(i+1, x_{i+1})$ with a straight line segment for every $i \in \mathbb{Z}$. We say that $x \in \mathbb{R}^{\mathbb{Z}}$ and $x^* \in \mathbb{R}^{\mathbb{Z}}$ cross if the Aubry graphs of x and

x^* intersect. There are two ways for two Aubry graphs to intersect: they either intersect at integer point i or intersect between i and $i+1$.

There is an action T of group $\mathbb{Z}^2$ on $\mathbb{R}^{\mathbb{Z}}$: if $(a,b) \in \mathbb{Z}^2$ and $x \in \mathbb{R}^{\mathbb{Z}}$, then

$$T_{(a,b)} = x^* \quad \text{where} \quad x_i^* = x_{i-a} + b$$

The action correspond to the translation of the Aubry graph by $(a,b) \in \mathbb{Z}$. It is easy to see that if $x \in \mathcal{M}$, then $T_{(a,b)}x \in M$ for any $(a,b) \in \mathbb{Z}$.

One of the fundamental properties of the action-minimizing orbits is that any two can only cross at most once.

Lemma 2.2 (Crossing Lemma). (1) *If $(x_j,\ldots,x_k)$ and $(x_j^*,\ldots,x_k^*)$ are minimal segments which are not equal, then the Aubry graph of $(x_j,\ldots,x_k)$ and $(x_j^*,\ldots,x_k^*)$ intersect at most once. Therefore, if $x \in \mathcal{M}$, $x^* \in \mathcal{M}$ and $x \neq x^*$, then the Aubry graph of x and x^* intersect at most at one point.*

(2) *Let $x \in \mathcal{M}$ be an action-minimizing trajectory. Then for any $(a,b) \in \mathbb{Z}^2$, x and $T_{(a,b)}x$ do not cross.*

(3) *If $x \in \mathcal{M}$ and $x^* \in \mathcal{M}$, x and x^* are ω-asymptotic or α-asymptotic, then x and x^* do not cross.*

Two orbits x and x^* are said to be ω-asymptotic (α-asymptotic, resp.) provided that $\lim_{i\to\infty}|x_i - x_i^*| = 0$ ($\lim_{i\to-\infty}|x_i - x_i^*| = 0$, resp.). The crossing lemma is an easy consequence of the transversality condition (H_4).

There is an action T of group $\mathbb{Z}^2$ on $\mathbb{R}^{\mathbb{Z}}$: if $(a,b) \in \mathbb{Z}^2$ and $x \in \mathbb{R}^{\mathbb{Z}}$, then

$$T_{(a,b)} = x^* \quad \text{where} \quad x_i^* = x_{i-a} + b$$

The action correspond to the translation of the Aubry graph by $(a,b) \in \mathbb{Z}$. It is easy to see that if $x \in \mathcal{M}$, then $T_{(a,b)}x \in M$ for any $(a,b) \in \mathbb{Z}$.

A point $x \in \mathbb{R}^{\mathbb{Z}}$ is said to be periodic with period $(q,p) \in (\mathbb{Z}\backslash\{0\}) \times \mathbb{Z}$ if $T_{(q,p)}x = x$. The existence of periodic minimal configuration for any period (q,p) can be established in the following way: For any periodic configuration $x \in \mathbb{R}^{\mathbb{Z}}$ with period (q,p), let $H_{q,p}(x) = H(x_0,x_1,\ldots,x_q)$. By properties (H_1) and (H_2) that $H_{q,p}$ attains its infimum at a point x^* on the set of all (q,p) periodic configurations. One needs to argue that x^* is a minimal configuration. The key is to show that any two minimum points of $H_{q,p}$ do not cross. For that one uses the property (H_3) and concludes that every segment of x^* of the length nq is minimal. This implies that $x^* \in \mathcal{M}$. We omit the details.

The fact that x^*, the minimizer for periodic sequences, is actually action-minimizing is a consequence of the order properties of the real line. The same result does not hold in high dimensions.

The most fundamental property of a minimal configuration is that for every $x \in \mathcal{M}$ there exists an orientation preserving circle homeomorphism ϕ and a lift of ϕ, $\tilde{\phi}$ such that $x_{i+1} = \tilde{\phi}(x_i)$. Let $\tilde{\alpha}(x) = \lim_{i\to\infty} x_i/i$. $\tilde{\alpha}(x)$ is called the rotation number of $x \in \mathcal{M}$. For any real number α, we let $\mathcal{M}_\alpha = \{x \in \mathcal{M} \mid \tilde{\alpha}(x) = \alpha\}$. The structure of $\mathcal{M}_\alpha$ is well understood:

1. $M_\alpha/\sim$ is non-empty and compact in $\mathbb{R}^{\mathbb{Z}}$ for any $\alpha \in \mathbb{R}$, where $x \sim y$ if there exists $(a,b) \in \mathbb{Z}^2$ such that $T_{(a,b)}x = y$.

2. If $\alpha = p/q \in \mathbb{Q}$, where q and p are relative primes, $\mathcal{M}_\alpha$ consists of periodic orbits of period (q,p) and the connecting orbits between any two consecutive periodic orbits. Let $x^1 \in M_{p/q}, x^2 \in M_{p/q}$ with $x^1 < x^2$ be two adjacent minimal periodic points. We define the following sets:

$$M^+(x^1,x^2) = \{x \in M_{p/q} | \alpha\text{-limit set of } x \text{ is } x^1 \text{ and } \omega\text{-limit set of } x \text{ is } x^2\}$$

$$M^-(x^1,x^2) = \{x \in M_{p/q} | \alpha\text{-limit set of } x \text{ is } x^2 \text{ and } \omega\text{-limit set of } x \text{ is } x^1\}$$

3. If $\alpha \notin \mathbb{Q}$ is irrational, then the whole set M_α can be embedded into a circle homeomorphism. i.e., there exists an orientation preserving circle homeomorphism ϕ and a lift $\tilde{\phi}$, whose rotation number is α, such that for any $x \in M_\alpha$, we have $x_{i+1} = \tilde{\phi}(x_i)$. There are two cases. The first one is that for every $a \in \mathbb{R}$, there is point $x \in M_\alpha$ such that $x_0 = a$. In this case, there is an invariant circle in the phase space with rotation number α. Conversely, if there is an invariant homotopically non-trivial circle in the phase space, then it can be shown that every orbit in the invariant circle is action-minimizing. The second case is that the projection of M_α to the circle is not onto. In this case, there is a unique recurrent invariant set on the circle for the circle map ϕ, by closeness of M_α, the projection of M_α must cover this invariant set. The corresponding orbits in the phase space is a Cantor set or a Cantorus, as is often called. In generic situations, all corresponding orbits for M_α, $\alpha \notin \mathbb{Q}$ are recurrent.

We need a refined classifications for the minimal trajectories with rational rotation numbers. For $\alpha = p/q$, we let $M_{(p/q)^0}$ be the set of minimal periodic orbits with period (q,p); We let

$$M_{(p/q)^+} = \bigcup_{(x^1,x^2)\text{adjacent pairs in } M_{(p/q)^0}} M^+(x^1,x^2)$$

and

$$M_{(p/q)^-} = \bigcup_{(x^1,x^2)\text{ adjacent pairs in } M_{(p/q)^0}} M^-(x^1,x^2)$$

In other words, we decomposed $M_{p/q}$ into three disjoint subsets: $M_{(p/q)^0}$, $M_{(p/q)^+}$ and $M_{(p/q)^-}$. Correspondingly, we let $\mathbb{R}'$ to denote the extension of the real numbers to replace p/q with p/q^-, p/q^0 and p/q^+. The set $\mathbb{R}'$ is totally ordered if we define, in addition to the usual order of $\mathbb{R}$, the new order: $p/q^- < p/q^0 < p/q^+$. The total order on $\mathbb{R}'$ gives rise to a unique topology on $\mathbb{R}'$.

3. Barrier functions and connecting orbits

In this section, we introduce our barrier functions for the twist maps. We then use the barrier function to give a simple proof to a remarkable theorem of Mather [5]. This section give some basic ideas involving barrier and will serve as a basis for our high dimensional construction.

To construct connecting orbits among various action-minimizing trajectories, Mather [5] introduced the idea of action-minimizing trajectories with constraints. In Mather's construction, these constraints are closed intervals of the real line $I_i, i \in \mathbb{Z}$ where one *a priori* requires that $x_i \in I_i$, for all $i \in \mathbb{Z}$. These constraint minimizers correspond to true trajectories in the phase space if the minimizers are contained in the interior of the constraints. The problem is then to construct proper constraints to guarantee that the minimal does not occur on the boundary of the constraints. The barrier functions, to be introduced next, turn out to be a useful tool in this regard.

Indeed, the so-called Peierl's barrier function and its variants play a very important role in the study of twist maps. Since the constraints we use will be somewhat different from that of Mather's, we introduce the barrier function in a very different way. Our barrier admits an easy generalization to high dimensions. Our constraints will consist of two different types. The first type is finite number of closed intervals I_{i_1}, I_{i_2}, $\ldots$, I_{i_k} for some positive integer k, where we, *a priori*, require $x_{i_j} \in I_{i_j}$ for $j = 1, \ldots, k$. The second type of constraints concerns asymptotic behavior. We typical requires that $x \in \mathbb{R}^{\mathbb{Z}}$ be α-asymptotic to some action-minimizing trajectory and ω-asymptotic to some other action-minimizing trajectories.

First, we need to introduce one-sided action-minimizing sequences, instead of the bi-infinite action-minimizing sequences. Again, a one sided sequence $x^+ = \{x_i\}_{i\in\mathbb{N}^+}$ or $x^- = \{x_i\}_{i\in\mathbb{N}^-}$ is said to be action-minimizing if every finite segment of the sequence is action-minimizing. Here $\mathbb{N}^+ = \{0, 1, 2, \ldots, \}$ and $\mathbb{N}^- = \{, \ldots, -2, -1, 0\}$. We define the rotation number of an action-minimizing (one-sided) sequence $\{x_i\}_{i\in\mathbb{N}^+}$ ($\{x_i\}_{i\in\mathbb{N}^-}$, resp.) by $\tilde{\alpha}^+(x^+) = \lim_{i\to\infty} x_i/i$ ($\tilde{\alpha}^-(x^-) = \lim_{i\to-\infty} x_i/i$, resp.). By the crossing lemma, such limit always exists. Moreover, for any rotation number $\alpha \in \mathbb{R}$ and any real number $a \in \mathbb{R}$, there are at least one minimizing $x^+ = \{x_i\}_{i\in\mathbb{N}^+}$ and one $x^- = \{x_i\}_{i\in\mathbb{N}^-}$ such that $x_0 = a$ and x^+, x^- both have rotation number α.

One may obtain x^+ (and similarly x^-) in the following way: let $\{x_i^N\}_{i=0}^N$ be the action-minimizing segment with $x_0^N = a$ and $x_N^N = a + N\alpha$. The sequence of segments x^N has a convergent subsequence x^+ in the sense that there is a sequence of positive integers n_k, $k = 1, 2, \ldots$, $n_k \to \infty$ as $k \to \infty$ such that $\lim_{k\to\infty} x_i^{n_k} = x_i^+$ for all $i = 0, 1, \ldots$. Clearly, x^+ is action-minimizing and has rotation number α.

In fact, if $\alpha \in \mathbb{Q}$ and there is no $x \in \mathcal{M}_{\alpha^0}$ with $x_0 = a$, then there are at least two one-sided positive sequences that start with a and have rotation number α (one going forward and one going backward). Similarly, there are at least two negative one-sided action-minimizing sequences.

The following proposition is an easy consequence of properties $H_1 - H_4$.

Proposition 3.1. *Let $\{x^a\}_{i=0}^{\infty}$ be a one-sided action-minimizing sequence with $x_0^a = a$ and rotation number α.*

1. If $\alpha \notin \mathbb{Q}$, then $\{x^a\}_{i=0}^{\infty}$ does not cross any trajectories in M_α and therefore, any two such choices of $\{x^a\}_{i=0}^{\infty}$ are ω-asymptotic.

2. If $\alpha = p/q \in \mathbb{Q}$, then $\{x^a\}_{i=0}^{\infty}$ does not cross any trajectories in $M_{(p/q)^0}$. If x^1, $x^2 \in M_{(p/q)^0}$ are adjacent pairs such that $x_0^1 < a < x^2$, then $\{x^a\}_{i=0}^{\infty}$ is ω-asymptotic either to x^1 or x^2.

The corresponding orbits in the phase space for one sided action-minimizing sequences represent the stable and unstable sets of the action-minimizing trajectories.

We remark that two one-sided sequences may cross both at $i = 0$ and at $\pm\infty$. This does not contradict the crossing lemma. However, this occurs only in some non-generic situations.

We first define barrier function for irrational rotation numbers.

Fix a rotation number $\alpha \in \mathbb{R}/\mathbb{Q}$ and fix a real number $a \in \mathbb{R}$. Let $x^a \in \mathbb{R}^{\mathbb{Z}}$ be a bi-infinite sequence satisfying the following properties

- $x_0^a = a$;
- $\{x_i^a\}_{i=0}^{\infty}$ is action-minimizing and the ω-limit set of x^a is in M_α;
- $\{x_i^a\}_{i=-\infty}^{0}$ is action-minimizing and the α-limit set of x^a is in M_α.

For twist maps, such x^a obviously exists, even though it may not necessarily be unique. In fact, if there is $x^\alpha \in M_\alpha$ with $x_0^\alpha = a$ then $x^a = x^\alpha$. If $x^\alpha, \bar{x}^\alpha \in M_\alpha$ with $x_0^\alpha \le a \le \bar{x}_0^\alpha$, then $x_i^\alpha \le x_i^a \le \bar{x}_i^\alpha$, for all $i \in \mathbb{Z}$. Now we can define the barrier function which depend on $\alpha \in \mathbb{R}/\mathbb{Q}$ and $a \in \mathbb{R}$:

$$P_\alpha(a) = \lim_{N\to\infty} \left((\sum_{i=-N}^{N-1} h(x_i^a, x_{i+1}^a)) - \inf_{\{x \mid x_{\pm N} = x_{\pm N}^a\}} (\sum_{i=-N}^{N-1} h(x_i, x_{i+1})) \right)$$

It is easy to see that for any $\alpha \in \mathbb{R}/\mathbb{Q}$ and $\alpha \in \mathbb{R}$, the limit in the above definition for $P_\alpha(a)$ exists (monotonic with respect to N) and the limit is nonnegative and bounded. Moreover, the value of $P_\alpha(a)$ is independent of the choice of x^a, even if x^a is not unique.

Now we define the barrier functions for rational rotation numbers. For any $a \in \mathbb{R}$ and $\alpha = (p/q)^+ \in \mathbb{R}'$, if there is a periodic minimal trajectory $x^{p/q}$ of period (q,p) such that $x_0^{p/q} = a$, then we let $P_{(p/q)^+}(a) = 0$, otherwise, there are two adjacent periodic trajectories $x^1, x^2 \in M_{(p/q)^0}$ of period (q,p) such that $x_0^1 < a < x_0^2$. Let $x^a \in \mathbb{R}^{\mathbb{Z}}$ be a bi-infinite sequence satisfying the following properties

- $x_0^a = a$;
- $\{x_i^a\}_{i=0}^{\infty}$ is action-minimizing and the ω-limit set of x^a is $x^2 \in M_{(p/q)^0}$;

- $\{x^a_i\}^0_{i=-\infty}$ is action-minimizing and the α-limit set of x^a is in $x^1 \in M_{(p/q)^0}$.

Now the barrier function for this case is defined in exactly the same way as before:

$$P_{(p/q)^+}(a) = \lim_{N\to\infty}\left((\sum_{i=-N}^{N-1} h(x^a_i, x^a_{i+1})) - \inf_{\{x \mid x_{\pm N} = x^a_{\pm N}\}} (\sum_{i=-N}^{N-1} h(x_i, x_{i+1}))\right)$$

Similarly, we can define the barrier functions for $\alpha = (p/q)^- \in \mathbb{R}'$, where we switch the α-limit and ω-limit set in the definition for $P_{(p/q)^+}(a)$.

For $\alpha = (p/q)^0$, the set $M_{(p/q)^0}$ is typically a disjoint union of countably many periodic trajectories. For any $a \in \mathbb{R}$, there are typically two neighboring periodic trajectories in $M_{(p/q)^0}$, we need to specify one of these two and then requires x^a to be both α and ω asymptotic to the chosen one. Thus the barrier function depends on the choice of left or right periodic trajectory. When needed, we will make explicit choice $x \in M_{(p/q)^0}$ and we will write the barrier function in this case as $P_x(a)$.

We list some of the simple properties for $P_\alpha(a)$:

- $P_\alpha(a+1) = P_\alpha(a)$ for any $a \in \mathbb{R}$ and $\alpha \in \mathbb{R}'$;
- $P_\alpha(a) \geq 0$ for any $a \in \mathbb{R}$ and $\alpha \in \mathbb{R}'$; $P_\alpha(a) = 0$ if and only there is a $x^\alpha \in M_\alpha$ such that $x^\alpha_0 = a$ and in that case, $x^a = x^\alpha$.
- $P_\alpha(a)$ is Lipschitz continuous with respect to a and it is continuous with respect to $\alpha \in \mathbb{R}'$ with the topology in $\mathbb{R}'$. However, it is not continuous with respect to α with the usual topology of $\mathbb{R}$.
- $P_\alpha(a) = 0$ for any $a \in \mathbb{R}$ if and only if there is an invariant curve in the phase space with rotation number α.

We remark that the order properties we used in the definitions are only valid for real numbers. When we go to high dimensions later in the paper, we will need to slightly revise the definition of the barrier function, but the basic ideas remain the same.

To use the barrier functions to construct the connecting orbits, we need to define the constraint barrier function. We only give the formal definition for irrational rotation numbers, the other cases are similar. Let $I \subset \mathbb{R}$ be a closed and finite interval of the real line, let $a \in I \subset \mathbb{R}$ and let $\alpha \in \mathbb{R}/\mathbb{Q}$. The constraint barrier function $P_\alpha(a, I)$ is a real number defined by:

$$P_\alpha(a, I) = \lim_{N\to\infty}\left((\sum_{i=-N}^{N-1} h(x^a_i, x^a_{i+1})) - \inf_{\{x \mid x_{\pm N} = x^a_{\pm N}; x_0 \in I\}} (\sum_{i=-N}^{N-1} h(x_i, x_{i+1}))\right),$$

where x^a is the same as before. The only difference is that the infimum in the above is taken for all $x \in \mathbb{R}^{\mathbb{Z}}$ with $x_0 \in I$.

The usefulness of the constraint barrier functions can be demonstrated by the following simple proposition.

Proposition 3.2. *Let $I \subset \mathbb{R}$ be a closed finite interval and let $\alpha \in \mathbb{R}'$ be a rotation number. Suppose $P_\alpha(b, I) > 0$ for $b \in \partial I$, where ∂I is the boundary of I. Then there is a constraint minimal trajectory $x \in \mathbb{R}^Z$ with x_0 in the interior of I and x is both α and ω asymptotic to trajectories in M_α.*

Proof. Let $a^* \in I$ be the point such that $P_\alpha(a^*, I) = 0$. Such point a always exists and can be found in the following way: take any $a \in I$ and let x^a be the configuration as in the definition of the barrier function. Let $\{x_i^N\}_{i=-N}^N$ be the action-minimizing segment with $x_{-N}^N = x_{-N}^a$, $x_N^N = x_N^a$ and $x_0 \in I$. Such minimizing segment always exists for each positive integer N. Let x^∞ be a sub-sequential limit of $\{x^N\}_{i=-N}^N$ as $N \to \infty$. Then $\{x^\infty\}_{i=-\infty}^0$ and $\{x^\infty\}_0^{i=\infty}$ are both action-minimizing. Let $a^* = x_0^\infty \in I$, then $P_\alpha(a^*, I) = 0$.

The fact that $P_\alpha(b, I) > 0$ for $b \in \partial I$ implies that a^* is in the interior of I. Therefore x^{a^*} is a stationary trajectory, which correspond to a true trajectory in the phase space.

The constraint action-minimizers may not necessarily be global action-minimizer. □

The finite closed interval in the Proposition 3.2 is called a *barrier* for rotation number α.

To connect action-minimizing trajectories with different rotation numbers, we define the joint barrier functions. The joint barriers are always defined with constraints.

Let α_1, α_2 be two rotation numbers in $\mathbb{R}'$ and let $I \subset \mathbb{R}$ be a finite closed interval and $a \in I$ be a real number. Again, let $x^a \in \mathbb{R}^\mathbb{Z}$ be a trajectory such that

- $x_0^a = a$;
- $\{x_i^a\}_{i=0}^\infty$ is action-minimizing and x^a is ω asymptotic to a trajectory in M_{α_2};
- $\{x_i^a\}_{i=-\infty}^0$ is action-minimizing and x^a is α asymptotic to a trajectory in M_{α_1}.

Such x^a always exists. We define

$$P_{\alpha_1,\alpha_2}(a, I) = \lim_{N\to\infty} \left(\left(\sum_{i=-N}^{N-1} h(x_i^a, x_{i+1}^a)\right) - \inf_{\{x \mid x_{\pm N} = x_{\pm N}^a; x_0 \in I\}} \left(\sum_{i=-N}^{N-1} h(x_i, x_{i+1})\right) \right)$$

Above limit always exist and is independent the choice of x^a. We remark that, without the constraint I, if $\alpha_1 \neq \alpha_2$ (as real numbers), then the above limit is always ∞.

Similar to Proposition 3.2, we have

Proposition 3.3. *Let $I \subset \mathbb{R}$ be a closed finite interval and let $\alpha_1, \alpha_2 \in \mathbb{R}'$ be rotation numbers. Suppose $P_{\alpha_1,\alpha_2}(b, I) > 0$ for $b \in \partial I$, where ∂I is the boundary of I. Then there is a constraint minimal trajectory $x \in \mathbb{R}^Z$ with x_0 in the interior of I and x is ω asymptotic to a trajectory in M_{α_2} and α asymptotic to a trajectory in M_{α_1}.*

Proof. Let $a^* \in I$ be a point such that $P_{\alpha_1,\alpha_2}(a^*, I) = 0$, then a^* is in the interior of I and x^{a^*} is the required stationary trajectory. Again, a^* is found by minimizing finite segment from $-N$ to N of any given configuration x^a, $a \in I$ with fixed end points and within the constraint I. Taking $N \to \infty$ to obtain a sub-sequential limit, which will be the desired action-minimizer. □

Definition 3.4. *Let $I \subset \mathbb{R}$ be a closed finite interval and let $\alpha_1, \alpha_2 \in \mathbb{R}'$ be rotation numbers. Suppose $P_{\alpha_1,\alpha_2}(b, I) > 0$ for $b \in \partial I$, then I is called a* joint barrier *or simply a* barrier *for the pair (α_1, α_2).*

The existence of joint barrier for any pair (α_1, α_2) implies the existence of a connecting orbits from M_{α_1} to M_{α_2}.

The following fact is very important: $P_{\alpha_1,\alpha_2}(a, I)$ is continuous with respect to α_1, α_2 with $\mathbb{R}'$ topology, continuous with respect to I and Lipschitz with respect to a. For $\alpha_1 = \alpha_2$, it is the usual constraint barrier function, $P_{\alpha_1,\alpha_2}(a, I) = P_{\alpha_1}(a, I)$.

Suppose that there is no invariant curve with a given rotation number $\alpha \in \mathbb{R}'$, then there exists $b_1, b_2 \in \mathbb{R}$, $b_1 < b_2$ (one may take $b_2 = b_1 + 1$) such that $P_\alpha(b_1) > 0$ and $P_\alpha(b_2) > 0$ and $P_\alpha(a) = 0$ for some $a \in I = [b_1, b_2]$. By the continuity of the joint barrier functions, if α_1 and α_2 are close enough to α in $\mathbb{R}'$ topology, then I is a joint barrier for the pair (α_1, α_2), i.e., $P_{\alpha_1,\alpha_2}(b, I) > 0$ for $b \in \partial I$. We have the following proposition.

Proposition 3.5.
(1) *Let $\alpha \in \mathbb{R}/\mathbb{Q}$ be an irrational rotation number. Suppose that there is no invariant curve with rotation number α, then there is an $\epsilon > 0$ such that if $|\alpha_1 - \alpha| < \epsilon$, $|\alpha_2 - \alpha| < \epsilon$, then there is a joint barrier for the pair (α_1, α_2) and hence there is a connecting orbit from M_{α_1} to M_{α_2}.*
(2) *Let $\alpha = (p/q)^+ \in \mathbb{R}'$ be a rotation number. Suppose that there is no invariant curve with rotation number $(p/q)^0$ or $(p/q)^+$, then there is an $\epsilon > 0$ such that if $0 < \alpha_1 - p/q < \epsilon$, then there is a joint barrier for the pair $((p/q)^+, \alpha_1)$ and hence there is a connecting orbit from $M_{(p/q)^0}$ to M_{α_1}.*
(3) *Let $\alpha = (p/q)^- \in \mathbb{R}'$ be a rotation number. Suppose that there is no invariant curve with rotation number $(p/q)^0$ or $(p/q)^-$, then there is an $\epsilon > 0$ such that if $0 < p/q - \alpha_1 < \epsilon$, then there is a joint barrier for the pair $(\alpha_1, (p/q)^-)$ and hence there is a connecting orbit from M_{α_1} to $M_{(p/q)^0}$.*

The above proposition provides the tools for connecting action-minimizing trajectories in a neighborhood of a broken invariant circle. To extend the connection to a longer distance, we need the following proposition. First, we deal only with irrational rotation numbers.

Proposition 3.6. *Let α_1, α_2 and α_3 be irrational numbers and let I_1 and I_2 be joint barriers for pairs (α_1, α_2) and (α_2, α_3) respectively. Then there exists a constraint minimal trajectory x satisfying the following properties:*

(1) *The α limit of x is in M_{α_1};*
(2) *The ω limit of x is in M_{α_3};*
(3) *x_0 is in the interior of I_1;*
(4) *x_N is in the interior of $I_2+[N\alpha_2]$ for some large N and where $[N\alpha_2]$ is the integer part of $N\alpha_2$.*

Proof. Fix a positive integer N. Consider trajectories with the following constraints:

(1) The α limit of x is in M_{α_1};
(2) The ω limit of x is in M_{α_3};
(3) $x_0 = a_1 \in I_1$;
(4) $x_N = a_2 \in I_2 + [N\alpha_2]$ for some large N, where again $[N\alpha_2]$ is the integer part of $N\alpha_2$.

It is easy to see that the constraint action-minimizer with the above constraints always exists. To prove the proposition, we need to show that for proper choice of N, the action-minimizer is actually in the interior of the constraints and thus give rise to true trajectories in the phase space.

To understand the constraint action-minimizer with the above constraints, we decomposed it into three independent segments. The minimizer for the segment $\{x_i\}_{i=-\infty}^{0}$ is exactly the same $\{x^{a_1}\}_{i=-\infty}^{0}$ as given in the definition of the barrier functions. Likewise, the segment $\{x_i\}_{i=N}^{\infty}$ is x^{a_2} with proper translations by integers. For the finite segment $\{x_i\}_{i=0}^{N}$, we need to study its asymptotic properties as $N \to \infty$.

Let $\{x_i^{(a_1,a_2,N)}\}_{i=0}^{N}$ be a finite configuration that minimizes the segment $\{x_i\}_{i=0}^{N}$ with $x_0 = a_1$ and $x_N = a_2$. For each $i \geq 0$, the sequence $x_i^{(a_1,a_2,N)}$ is bounded for all $N > i$, and therefore there is a convergent subsequence. Let $\{x_i^{(a_1,a_2)}\}_{i=0}^{\infty}$ be a sub-sequential limit of $\{x_i^{(a_1,a_2,N)}\}_{i=0}^{N}$ as $N \to \infty$.

$\{x_i^{(a_1,a_2)}\}_{i=0}^{\infty}$ is a one-sided action-minimizing sequence, as $\{x_i^{(a_1,a_2,N)}\}_{i=0}^{N}$ is action-minimizing for every N. $\{x_i^{(a_1,a_2)}\}_{i=0}^{\infty}$ has α_2 as its rotation number. Moreover, the ω-limit set of $\{x_i^{(a_1,a_2)}\}_{i=0}^{\infty}$ consists action-minimizing trajectories with rotation number α_2.

To get the limit behavior of $\{x_i^{(a_1,a_2,N)}\}_{i=0}^{N}$ from the side on $I_2 + [N\alpha_2]$, we consider the translation of the segment by $T_{(-N,-[N\alpha_2])}$ where $[N\alpha_2]$ is the integer part of the real number $N\alpha_2$. For each $i \leq 0$, the sequence $x_{i+N}^{(a_1,a+2,N)} - [N\alpha_2]$ is bounded for all $N < -i$, and therefore there is a convergent subsequence. We let $\{x_i^{(a_1,a_2)}\}_{i=-\infty}^{0}$ be a sub-sequential limit of $\{x_{i+N}^{(a_1,a_2,N)} - [N\alpha_2]\}_{i=-N}^{0}$ as $N \to \infty$. Again, this limit is a one-sided action-minimizing sequence with rotation number α_2.

Let $x_i^{a_1} = x_i^{(a_1,a_2)}$ for $i \geq 0$ and let

$$\delta = \min_{a_1 \in \partial I_1} P_{\alpha_1,\alpha_2}(a_1, I_1) > 0$$

Recall that

$$P_{\alpha_1,\alpha_2}(a_1, I_1) = \lim_{n\to\infty}\left(\sum_{i=-n}^{n-1} h(x_i^{a_1}, x_{i+1}^{a_1}) - \inf_{\{x \mid x_{\pm n}=x_{\pm n}^{a_1};x_0\in I_1\}}(\sum_{i=-n}^{n-1} h(x_i, x_{i+1}))\right)$$

We may choose a positive integer N_1 large enough such that if $n \geq N_1$,

$$\sum_{i=-n}^{n-1} h(x_i^{a_1}, x_{i+1}^{a_1}) - \inf_{\{x \mid x_{\pm n}=x_{\pm n}^{a_1};x_0\in I_1\}}(\sum_{i=-n}^{n-1} h(x_i, x_{i+1})) > \frac{3}{4}\delta$$

for all $a_1 \in \partial I_1$.

$\{x_i^{a_1}\}_{i\geq 0}$ is a sub-sequential limit of $\{x_i^{(a_1,a_2,N)}\}_{i=0}^{N}$. We may assume, without loss of generality, that it is actually the limit. There exists a positive integer $N_2 > N_1$ such that for $N > N_2$

$$|\sum_{i=0}^{N_1-1}(h(x_i^{(a_1,a_2,N)}, x_{i+1}^{(a_1,a_2,N)}) - h(x_i^{a_1}, x_{i+1}^{a_1}))| < \frac{1}{4}\delta$$

for all $a_1 \in \partial I_1$.

Combining the above two inequalities, we see that for any $a_2 \in I_2 + [N\alpha_2]$ and $N > N_2$, the boundary points of I_1 can not be in the constraint action-minimizers. Action for trajectories passes through the boundary can be reduced by at least the amount of $\frac{1}{2}\delta$ within the constraints.

The boundary points of $I_2 + [N\alpha_2]$ can be considered in the similar way, except that translations have to be made at each step so as to bring the interval to bounded domain. We omit the details.

This proves the proposition. $\square$

The following proposition can be proved in the same way:

Proposition 3.7. *Let α_1, α_2, ..., α_k be irrational numbers and let I_1, I_2, ..., I_{k-1} be joint barriers for (α_1, α_2), (α_2, α_3), ..., (α_{k-1}, α_k) respectively. Then there exists a constraint minimal trajectory x and integers $N_1 = 0 < N_2 < \ldots < N_{k-1}$ such that*

1. *The α limit of x is in M_{α_1};*
2. *The ω limit of x is in M_{α_k};*
3. *x_{N_1} is in the interior of I_1;*
4. *x_{N_i} is in the interior of $I_i + [(N_i - N_{i-1})\alpha_i]$ for all $i = 2, 3, \ldots, k-1$.*

We have so far restricted ourselves to connecting minimal trajectories with irrational rotation numbers. We now turn our attention to connecting orbits near rational rotation numbers. The difficulty here is that for these cases, there are two possible choices of asymptotic behaviors which require special care.

Fix two relatively prime numbers (q, p). Let $x^1 < x^2$ be two adjacent minimal periodic trajectories with period (q, p). Let $I_1 \subset (x_0^1, x_0^2)$ and $I_2 \subset (x_0^1, x_0^2)$ be barriers for $(p/q)^-$ and $(p/q)^+$ respectively. We remark that if there is no invariant curve with rotation number (q,p) in the phase space, then such pair $x^1 < x^2$ and barriers I_1, I_2 can always be found.

By continuity of the barrier functions, I_1 is also a joint barrier for $(\alpha_1, (p/q)^-)$ and I_2 is a joint barrier for $((p/q)^+, \alpha_2)$ for some $\alpha_1 < (p/q)^- < (p/q)^+ < \alpha_2$ with $(p/q) - \alpha_1$ and $\alpha_2 - (p/q)$ small.

The next proposition show that one can connect M_{α_1} with M_{α_2} with a constraint action-minimizer.

Proposition 3.8. *There is a constraint action-minimizer x with the following properties:*

(1) *x is α-asymptotic to a trajectory $x_1^\alpha \in M_{\alpha_1}$;*
(2) *x is ω-asymptotic to a trajectory $x_2^\alpha \in M_{\alpha_2}$;*
(3) *x_0 is in the interior of I_1;*
(4) *x_{qN} is in the interior of $I_2 + Np$ for some large N;*
(5) *x_{qK} is in the interior of $[x_0^1 + Kp, x_0^1 + Kp + \eta]$ for some $\eta > 0$ and some positive integer $K < N$.*

Proof. The idea of the proof is similar to that of the previous proposition with irrational rotation numbers. One of the differences is that we have a fifth condition, adding more constraints to the minimizer. The addition of the fifth constraint will become clear later. Fix positive integer $K = [N/2] < N$ and positive number $\eta > 0$, we consider the set of all trajectories x with the following constraints:

(1) x is α-asymptotic to a trajectory $x_1^\alpha \in M_{\alpha_1}$;
(2) x is ω-asymptotic to a trajectory $x_2^\alpha \in M_{\alpha_2}$;
(3) x_0 is in I_1;
(4) x_{qN} is in $I_2 + Np$;
(5) x_{qK} is in the interval $[x_0^1 + Kp, x_0^1 + Kp + \eta]$.

The action-minimizer for trajectories with above constraints can always be found. To prove the proposition, we need to show that for proper choice of N, K and δ, the constraint action-minimizer is actually in the interior of the constraints, as oppose to being on the boundary.

Again, we focus our attention to the segment of the minimizer from $i = 0$ to $i = Nq$. First, we drop the fifth constraint and study the limit behavior of the minimizer as $N \to \infty$. Fix $a_1 \in I_1$ and $a_2 \in I_2 + Np$ and let $\{x_i^{(a_1,a_2,N)}\}_{i=0}^{Nq}$ be a action-minimizer with $x_0 = a_1$ and $x_{Nq} = a_2$. As $N \to \infty$, The sequence $\{x_i^{(a_1,a_2,N)}\}_{i=0}^{Nq}$ has a sub-sequential limit, say $\{x_i^{(a_1,a_2)}\}_{i=0}^{\infty}$, which is a one-sided action-minimizer with rotation number p/q. There are two distinct possible asymptotic behaviors for $\{x_i^{(a_1,a_2)}\}_{i=0}^{\infty}$: one being that it is asymptotic to x^1 and the other one being that it is asymptotic to x^2. The first case is what we want, while the second case will not give us the desired barrier estimate. The addition of the fifth constraint is to guarantee the first case always occurs.

Fix a real number $\eta > 0$ and a positive integer $K < N$. Consider configurations $\{x_i\}_{i=0}^{Kq}$ such that $x_0 = a_1$, $x_{Kq}^1 \leq x_{Kq} \leq x_{Kq}^1 + \eta$ and

$x_{Nq} \in a_2 \in I_2 + Nq$ We claim that for η sufficiently small and K sufficiently large, $\{x_i^{(a_1,a_2)}\}_{i=0}^{\infty}$ is asymptotic to x^1. Suppose that this is not true, i.e, $\{x_i^{(a_1,a_2)}\}_{i=0}^{\infty}$ is asymptotic to x^2. We first note that the one-sided action minimizer $\{x_i^{(a_1,a_2)}\}_{i=0}^{\infty}$ can also be regarded as a sub-sequential limit as $K \to \infty$, regardless the constraint at x_{Nq}. Let $x^- \in M_{(p/q)^-}$ be an action-minimizer such that $x_0^2 < x_0^- < a_1$. Then x^- and $\{x_i^{(a_1,a_2)}\}_{i=0}^{\infty}$ cross exactly once, say, at or around $i = J$. By choosing K large, we obtain an action-minimizing segment $\{x_i\}_{i=0}^{Kq}$ arbitrarily close to $\{x_i^{(a_1,a_2)}\}_{i=0}^{\infty}$ for $i = 0, 1, \ldots, J+1$ such that $x_0 = a_1$, $0 < x^2_- x_j < \delta$ and $0 < x_{Kq} - x_{Kq}^1 < \eta$. This action-minimizing segment crosses x^- at or around $i = J < Kq$. Since they can't cross more than once, we also have $|x_{Kq} - x_{Kq}^-| \leq \delta$. For δ small is is impossible by the transversality conditions. This proves the claim.

We conclude that $\{x_i^{(a_1,a_2)}\}_{i=0}^{\infty}$ is asymptotic to x^1.

The argument for the side from $I_2 + Nq$ is similar. First we study the asymptotic behavior of the minimal segments $\{x_i^{(a_1,a_2,N)}\}_{i=0}^{Nq}$ translated by the action $T_{(-Nq,-p)}$ and let $\{x_i^{(a_1,a_2)}\}_{i=-\infty}^{0}$ be a subsequential limit. The additional constraint, with η being small, guarantees that it is α-asymptotic to x^1 instead of x^2.

By the definitions of our barrier functions, for large N, and hence large $k = [N/2]$, and small δ, the constraint minimizer is contained in the interior of I_1 and $I_2 + Nq$. It remains to show that $x_{Kq}^{(a_1,a_2,N)} \neq x_{Kq}^1 + \delta$. This is true because the sequence is asymptotic to x^1 from both sides and the barrier function $P_{x^1}(x_{Kq}^1 + \delta) = P_{x^1}(x_0^1 + \delta) > 0$.

This proves the proposition. □

Let $(\omega_1, \omega_2) \subset \mathbb{R}$ be an interval in the real line. Suppose that there is no invariant curve with rotation number α, $\alpha \in (\omega_1, \omega_2)$. The propositions of this section proves the following theorem, which are the main results of Mather [5].

Theorem 3.9 (Mather). *Suppose $\omega_1 < \alpha_1, \alpha_2 < \omega_2$. Then there is a trajectory in the phase space whose α-limit set lies in M_{α_1} and whose ω-limit set lies in M_{α_2}.*

Moreover, for each $i \in \mathbb{Z}$, a real number $\omega_1 \leq \alpha_i \leq \omega_2$ and a positive number ϵ_i, there exists an orbit in the phase space $(\ldots, P_j, \ldots)$ and an increasing bi-infinite sequence of integers $j(i)$ such that distance between $P_{j(i)}$ and M_{α_i} is smaller than ϵ_i.

4. Lagrangian flow

In this section, we extend the techniques developed in the last section to higher dimensions and generalize Mather's theorem (Theorem 3.9) to the so-called *a priori* systems. In higher dimensions, a natural generalization of the twist maps is the positive definite Lagrangian flow.

We consider Lagrangian systems on a compact manifold M. For most of this paper, this manifold will be $\mathbb{T}^2$. Let $L : TM \times \mathbb{R} \to \mathbb{R}$ be a C^2 function such that $L = L(x, \dot{x}, t) = L(x, \dot{x}, t+1)$ is 1-periodic in t. L is called a Lagrangian on M.

Let γ be an absolutely continuous curve in M. The basic principle in Lagrangian formulation of the classical mechanics is that, if γ is a true trajectory of the mechanical system, then it must be an extremal of the action, i.e., a trajectory $\gamma(t)$ must satisfy the following fixed endpoints variational equation:

$$\delta \int_a^b L(\gamma(t), \gamma'(t), t)dt = 0$$

In local coordinates, trajectories are solutions of the following Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial v}(x, \dot{x}, t) = \frac{\partial L}{\partial x}(x, \dot{x}, t)$$

We need to make the following three conventional hypothesis for L:

- *Positive definiteness*: For each $x \in M$, $v \in T_xM$ and $t \in \mathbb{T}$, the restriction of L to $TM_x \times t$ is strictly convex in the sense that its Hessian second derivative $(\partial^2 L(x, v, t)/\partial v_i \partial v_j)$ is everywhere positive definite.
- *Super-linear Growth*:

$$\frac{L(x, v, t)}{\|v\|} \to +\infty, \quad \text{as} \quad \|v\| \to \infty$$

  where $\| \; \|$ is a Riemannian metric on M and x ranges over M and $t \in \mathbb{T}$.
- *Completeness*: All the solutions of the Euler-Lagrange equation can be extended for all $t \in \mathbb{R}$.

Instead of considering absolutely continuous curve $\gamma : \mathbb{R} \to M$, it is often convenient to consider discrete sequences in M. From now on, we assume $M = \mathbb{T}^2$, whose universal covering space is $\mathbb{R}^2$. We consider the space $(\mathbb{R}^2)^Z = \mathbb{R}^{2\mathbb{Z}} = \{z \mid z : \mathbb{Z} \to \mathbb{R}^2\}$ of bi-infinite sequence of points in $\mathbb{R}^2$. An element $z \in \mathbb{R}^{2\mathbb{Z}}$ is also denoted by $z = \{z_i = (x_i, y_i)\}_{i \in \mathbb{Z}}$ and will again be be called an orbit or an trajectory. We use the product topology on $\mathbb{R}^{2\mathbb{Z}}$. i.e., convergence of a sequence $z^n \in \mathbb{R}^{2\mathbb{Z}}$ to $z \in \mathbb{R}^{2\mathbb{Z}}$ means that $\lim_{i \to \infty} z_i^n \to z_i$ for all $i \in \mathbb{Z}$.

For any two points $z_1, z_2 \in \mathbb{R}^2$, we define a function $h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$h(z_1, z_2) = \inf_{\gamma} \int_{t=0}^1 L(\gamma(t), \gamma'(t), t)dt$$

The infimum is taken among all absolutely continuous curves γ with $\gamma(0) = z_1$ and $\gamma(1) = z_2$. The infimum can always be achieved (Mather [4]).

We extend h to arbitrary finite segments $(z_j, \dots, z_k)$, $j < k$, of trajectories $z \in \mathbb{R}^{\mathbb{Z}}$ by

$$h(z_j, \dots, z_k) = \sum_{i=j}^{k-1} h(z_i, z_{i+1}).$$

We similarly define minimal or action-minimizing trajectories with respect to h. The set of all action-minimizing trajectories will be denoted by $\mathcal{M}(h)$ or simply $\mathcal{M}$.

In addition to the properties inherited from the Lagrangian L, h is also assumed to satisfy the following conditions $H_a - H_c$:

(H_a). "Invariance Condition": let $\{z_i = (x_i, y_i)\}_{i=j}^{k}$ be an action-minimizing segment. If $y_j = y_k = 0$, then $y_i = 0$ for all $i = j, \dots, k$. Let $h_1(x_1, x_2) = h((x_1, 0), (x_2, 0))$. Then h_1 satisfies conditions $H_1 - H_4$ of the twist maps. Moreover, we assume that every action-minimizing trajectory or segments for h_1 is action-minimizing for h.

We denote the set of all action-minimizers of h_1 by M^0. The action-minimizers in M^0 all have their second coordinate zero. We also denote the translations of M^0 by $T_{((0,i),(0,0))}$ as $M^i = T_{((0,i),(0,0))} M^0$. Its elements are all action-minimizers.

(H_b). "Uniqueness condition": there exists a positive number $\eta_2 > 0$ such that the only action-minimizing recurrent trajectories with the constraints $-\eta \le y_i \le 1 + \eta$, for all $i \in \mathbb{Z}$ are the ones given by h_1 and its translation $M^1 = T_{((0,1),(0,0))} M^0$.

(H_c). "Limit condition": there exists a real number $\delta > 0$, such that the following is true: for any point $a = (a_1, a_2) \in \mathbb{R}^2$ and a sequence of points $b^k = (b_1^k, b_2^k) \in \mathbb{R}^2$ with $0 < a_2 < 1$, $|b_2^k - 1| \le \delta$ and $|b_1^k / k| < B$, for some $B > 0$, $k \in \mathbb{N}$, then there is an integer $N \in \mathbb{N}$, independent of k such that for any $k \in \mathbb{N}$, the action-minimizing segment $z_0^k, \dots, z_k^k$ satisfying $z_0^k = a$, $z_k^k = b^k$ is all but at most N points inside the set $\mathbb{R} \times (1 - \delta, 1 + \delta)$.

Similarly, for any point $a = (a_1, a_2) \in \mathbb{R}^2$ and a sequence of points $b^k = (b_1^k, b_2^k) \in \mathbb{R}^2$ with $0 < a_2 < 1$, $|b_2^k| \le \delta$ and $|b_1^k / k| < B$, for some $B > 0$, $k = -1, -2, \dots$, then there is an integer $N \in \mathbb{N}$, independent of k such that for any $k = -1, -2, \dots$, the action-minimizing segment $z_k^k, \dots, z_{-1}^k, z_0^k$ satisfying $z_k^k = b^k$, $z_0^k = a$ is all but at most N points inside the set $\mathbb{R} \times (1 - \delta, 1 + \delta)$

As we will see later, all of the above conditions can be somewhat relaxed. For the invariance condition, M^0 does not have to be exactly on the the set with $y_i = 0$, $i \in \mathbb{Z}$, as we will see later. The uniqueness condition implies that, modulus translations, there is only one such invariant set M^0. The case with finite numbers of such invariant set can be dealt similarly. The third condition is a normal hyperbolicity condition and it probably is a consequence of the first two conditions and hence redundant. However, we are not making an effort to prove it. For the cases we study, these conditions can be easily verified.

Our first step is to connect action-minimizing trajectories from M^0 to M^1. Our barrier functions defined later will be based on these connections. For any point $a = (a_1, a_2) \in \mathbb{R}^2$ with $0 < a_2 < 1$ and a fixed real number α, we let $z^{a,\alpha} = \{z_i^{a,\alpha}\}_{i=-\infty}^{\infty} \in \mathbb{R}^{2\mathbb{Z}}$ be a trajectory such that

(1) $z_0^{a,\alpha} = a$ and both $\{z_i^{a,\alpha}\}_{i=0}^{\infty}$ and $\{z_i^{a,\alpha}\}_{i=-\infty}^{0}$ are action-minimizing;
(2) the ω-limit set of $z^{a,\alpha}$ is in $M_\alpha^1 \subset M^1$;
(3) the α-limit set of $z^{a,\alpha}$ is in $M_\alpha^0 \subset M^0$.

We prove the following lemma.

Lemma 4.1. *For any $a = (a_1, a_2) \in \mathbb{R}^2$, $0 < a_2 < 1$ and any real number α, there exist a trajectory $z^{a,\alpha}$ satisfying the above three conditions.*

Proof. For any positive integer N, let $\{z_i^N\}_{i=0}^N$ be an action-minimizing segment such that $z_0^N = a$ and $z_N^N = (a_1 + N\alpha, 1)$. It is easy to see that, by super-linear growth property of L, for each integer $i > 0$, the set $\{z_i^N \mid N \geq i\}$ is bounded. Therefore, $\{z_i^N\}_{i=0}^N$ has a sub-sequential limit $\{z_i^N\}_{i=0}^N$ as $N \to \infty$. Let $\{z^\infty\}_{i=0}^\infty$ be such a limit. Clearly, $\{z^\infty\}_{i=0}^\infty$ is action-minimizing and $z_0^\infty = a$. We need to show that the ω-limit set of $\{z^\infty\}_{i=0}^\infty$ is contained in M_α^1. By the limit condition (H_c), the limit is contained in $\mathbb{R} \times (1 - \delta, a + \delta)$. By the invariance condition (H_a), the limit is contained in M^1. The fact that it is contained in M_α^1 is again due to the super-linear growth property of L. $\square$

The trajectory $z^{a,\alpha}$ is constraint, so it may not neccessarily be a true trajectory. The following propositions shows the existence of connecting orbits between M^0 and M^1.

Proposition 4.2. *Let $\alpha \in \mathbb{R}$ be a real number and let M_α^0 and M_α^1 respectively be the sets of action-minimizing trajectories in M^0 and M^1 with rotation number α. There exists an action-minimizing trajectory whose α limit set is contained in M_α^0 and whose ω limit set is contained in M^1.*

Proof. First we fix $\alpha \in \mathbb{Q}$ be a rational number. Let $x^\alpha \in M^0$ be an action-minimizing trajectory in M^0 with rotation number α such that $|x_0^\alpha| \leq 1$. Let $z^N = \{z_i^N\}_{i=-N}^N$ be an action-minimizing segments such that $z_{-N}^N = (x_{-N}^\alpha, 0)$ and $z^N = (x_N^\alpha, 1)$. We consider the limit of z^N as $N \to \infty$. We first observe that, by the super-linear growth property of L, there is a positive number M such that $|z_i^N - i\alpha| \leq M$ for all $N > 0$ and all i, $-N \leq i \leq N$. i.e., the first components of the action-minimizing segments are uniformly bounded from the straight line through the origin with slope α.

Again let $z_i^N = (x_i^N, y_i^N)$. For each $N \in \mathbb{N}$, let $j(N)$ be the integer with $-N \leq j(N) \leq N$ such that $y_{j(N)}^N \geq 1/2$ and $y_i^N < 1/2$ for all $i = -N, -N+1, \ldots, j(N) - 1$. Consider the sequence of translated segments $T_{(-j(N),0),(-[j(N)\alpha],0)}x^N$, where T is the $\mathbb{Z}^2 \times \mathbb{Z}^2$ action on $\mathbb{R}^{2\mathbb{Z}}$ and $[j(N)\alpha]$ is the integer part of $j(N)\alpha$. For each i, $-N - j(N) \leq i \leq N - j(N)$, $(T_{(-j(N),0),(-[j(N)\alpha],0)}x^N)_i$ is uniformly bounded by $\alpha i + M + 1$. Therefore, the sequence of segments has a convergent subsequence as $N \to \infty$. Let

$w = (u, v)$ be a sub-sequential limit, where u and v are sequences of real numbers. w is action-minimizing. It is either one-sided infinite sequence or bi-infinite sequence. By our choice of the sequence, $v_0 \geq 1/2$ and $v_i \leq 1/2$ for all $i < 0$. If w_i exists for all $i < 0$, then α limit set of w is contained in M^0_α. Similarly, if w_i exists for all $i \geq 0$, then the ω limit set is contained in M^1_α, by the limit condition. We claim that w is a bi-infinite sequence, i.e., w_i exists for all $i \in \mathbb{Z}$. Suppose that this is not true and, without loss of generality, suppose that w_i exists only for $i \leq n$ for some $n \in \mathbb{N}$. It is easy to see that $v_n = 1$ and there exists a (q, p) periodic trajectory $\bar{x}^\alpha \in M^1_\alpha$ such that $\bar{x}^\alpha_n = v_n$. Now we consider the segment $\{\bar{w}_i\}^n_{i=-m}$ for some large integer m such that $\bar{w}_i = w_{i+p} - (q, 0)$ for $i = -m, -m+1, \ldots, n-p$ and $\bar{w}_i = (\bar{x}^\alpha_{i+q}, 1) - (q, 0)$ for $i = n-p, \ldots, n$. Since the segment $\{\bar{w}_i\}^n_{i=n-q-1}$ is not action-minimizing, there exists $\eta > 0$ such that one can reduce the action of the segment, fixing the two endpoints, by an amount of at least δ. By taking m large, one can make $|\bar{w}_{-m} - w_{-m}|$ arbitrarily small and hence one can make

$$|h(\bar{w}_{-m}, \bar{w}_{-m+1}, \ldots, \bar{w}_n) - h(w_{-m}, w_{-m+1}, \ldots, w_n)| < \delta.$$

This contradicts to the fact that w is action-minimizing. This contradiction shows that w is a bi-infinite action-minimizing sequence which connects M^0_α to M^1_α for all rational number α.

For irrational α, one takes a sequence of connecting orbits constructed above and then take a sub-sequential limit.

This proves the lemma. □

The fundamental difference between the high dimensional cases and the twist maps is that now the set of points of the type $z^{a,\alpha}$ in Lemma 4.1 may be unbounded and each orbit $z^{a,\alpha}$ has a different asymptotic behavior even though they have the same rotation number. Recall that in the twist map case, if α is irrational, all trajectories in $A^{a,\alpha}$ has exactly the same asymptotic behavior and for α rational, there are only two distinct asymptotic behaviors. In fact, it is easy to show that for any fixed α, if there is no invariant curve with rotation number α, then the set of the points $z^{a,\alpha}$ is unbounded. Likewise, there are infinitely many distinct connecting orbits between M^0_α to M^1_α.

The difficulty is created by the fact that one may have a action-minimizing trajectory asymptotic to some $x^\alpha \in M^1_\alpha$ and another one, both starting from the same point, asymptotic to $x^\alpha + (k, 0)$ for some integer $k \neq 0$. In the phase space, these two trajectories have the same asymptotic limit, but with different phase or timing. But in the covering space, they are different trajectories.

To overcome this ambiguity and put some definiteness into the asymptotic behavior, we follow Mather [6] to introduce the concept of c-minimizing trajectories. The idea is to choose and fix one action-minimizing trajectory with some additional properties. These additional properties are, in some sense, to do a second minimization procedure and to minimize action-minimizing trajectories. Once this is achieved, we can extend our barrier functions for twist maps to the higher dimensional situations.

5. c-MINIMAL TRAJECTORIES

We go back to a positive Lagrangian system defined on a compact Riemannian manifold M. Assume that the Lagrangian $L : TM \times \mathbb{T} \to \mathbb{R}$ the conditions on positive definiteness, super-linear growth and completeness of the Euler-Lagrange flow.

Let ϕ_L be the Euler-Lagrange flow generated by L. Let μ be a ϕ_L invariant probability measure on $TM \times \mathbb{T}$. We define the average action of μ by

$$A(\mu) = \int L d\mu$$

Since L is bounded below, therefore the above integral is well-defined, although it may be $+\infty$. If $A(\mu) < \infty$, we may define the *rotation vector* for μ, $\rho(\mu) \in H_1(M, \mathbb{R})$. Let $c \in H^1(M, \mathbb{R})$ be a cohomology class and let λ_c be a closed 1-form on M whose cohomology class is c. λ_c can be regraded as a real-valued linear function on TM and therefore can be pilled back as a linear function on $TM \times \mathbb{T}$. The rotation vector $\rho(\mu)$ is uniquely characterized by the following equation:

$$< c, \rho(\mu) >= \int \lambda_c d\mu$$

The bracket on the left is the canonical pairing of $H^1(M, \mathbb{R})$ and $H_1(M, \mathbb{R})$. The fact that λ_c is μ integrable follows from $A(\mu) < \infty$ and L is super-linear. The fact that the right hand side is independent the choice of λ_c is elementary (cf. Mather [4]).

Let $U_L = \{(\rho(\mu), A(\mu)) \mid \mu \in \mathcal{M}\}$ where $\mathcal{M}$ is the set of all ϕ_L invariant probability measures with finite action. Since $\mathcal{M}$ is a convex set and both ρ and A are linear functions, U_L is a convex subset in the finite dimensional vector space $H_1(M, \mathbb{R}) \times \mathbb{R}$. Moreover, the projection of U_L on $H_1(M, \mathbb{R})$ is surjective (Mather [6]). It follows from the fact that L is bounded below, U_L is the epigraph of a convex function $\beta = \beta_L : H_1(M, \mathbb{R}) \to \mathbb{R}$. i.e., $U_L = \{(h, a) \in H_1(M, \mathbb{R}) \times \mathbb{R} \mid a \geq \beta(h)\}$.

It is easy to see that β has super-linear growth.

For any $h \in H_1(M, \mathbb{R})$, we call $\beta(h)$ the minimal average action of the rotation vector h. For any invariant probability measure μ such that $A(\mu) = \beta(\rho(\mu))$, we call μ a Mather measure or minimal measure or action-minimizing measure. Clearly, for every rotation vector $h \in H_1(M, \mathbb{R})$, there exists at least one Mather measure with h as its rotation vector.

Let $\alpha : H^1(M, \mathbb{R}) \to \mathbb{R}$ be the conjugate of β in the sense of convex analysis. In other words,

$$\alpha(c) = max\{< c, h > -\beta \mid h \in H_1(M, \mathbb{R})\},$$

for any $c \in H^1(M,\mathbb{R})$. Since β has super-linear growth, the maximum is always achieved and α is well-defined. α also has super-linear growth.

Let $A_c(\mu) = A(\mu) - < c, \rho(\mu) >$ for any $c \in H^1(M,\mathbb{R})$ and any invariant probability measure μ. We say μ is c-minimal if it minimizes A_c over all ϕ_L invariant probability measures. By the duality between $c \in H^1(M,\mathbb{R})$ and $h \in H_1(M,\mathbb{R})$, every Mather measure is c-minimal for some $c \in H^1(M,\mathbb{R})$ and every c-minimal measure is a Mather measure.

Let $M^c \subset TM \times \mathbb{T}$ be the support of all c-minimal measures for some fixed $c \in H^1(M,\mathbb{R})$. An important result of Mather is that the projection of M^c to $M \times \mathbb{T}$ is injective and M^c is a Lipschitz graph over the image of this projection.

It is an elementary excise to show that the Euler-Lagrange flows generated by L and $L - \lambda$ are exactly the same for any closed 1-form λ on M. By the definitions of rotation vectors and the average action, we re-write:

$$A_c(\mu) = \int (L - \lambda_c) d\mu.$$

Therefore, the c-minimal measures for L correspond to the absolute action-minimizing measure for the Lagrangian $L - \lambda_c$.

We modify $h(z_1, z_2)$ to define so-called c-generating function $h_c : M \times M \to \mathbb{R}$. We let

$$h_c(z_1, z_2) = \min \int_0^1 (L - \lambda_c)(d\gamma(t), t) dt - \alpha(c),$$

where $\alpha(c)$, as defined before, is the convex conjugate of the β function, and the minimum is taking over all curves $\gamma : [0,1] \to M$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. Note that h_c depends on the choice of λ_c. If $\lambda'_c = \lambda_c + du$ for some smooth function $u : M \to \mathbb{R}$, then $h'_c(z_1, z_2) = h_c(z_1, z_2) + u(z_2) - u(z_1)$. As we will see later, this dependence on the choice of λ_c will not affect our definition of c-minimal trajectories.

A finite segments of points $(z_j, \ldots, z_k)$, $z_i \in M$, for $i = j, \ldots, k$, is said to be c-minimal if

$$h_c(z_j, \ldots, z_k) \leq h_c(z'_l, \ldots, z'_m)$$

for any other segment $(z'_l, \ldots, z'_m)$ such that $z_j = z'_l$ and $z_k = z'_m$. Note that we allow the minimization procedure to change the length of segment.

A bi-infinite sequence of points in M, $(\ldots, z_i, \ldots)$ is said to be c-minimal if every finite segment of c-minimal.

There are two fundamental differences between c-minimal and action-minimizing we defined before. The first one is that c-minimal configurations are in M, while the action-minimizing trajectories we had before are in the covering space of M. The second one is that for c-minimal trajectory, one can not reduce the action by adding deleting points in the trajectory. Clearly, every c-minimal trajectory is action-minimizing if one lifts it to the covering space. However, the converse is not true, i.e., not every action-minimizing trajectory projects to a c-minimal trajectory for some c.

Let $\tilde{M}_c \subset TM \times \mathbb{T}$ be the support of all c-minimal Mather measures, it is easy to show that all trajectories in $\tilde{M}_c$ for Euler-Lagrange flow are c-minimal (Mather [4]). An important result of Mather [4] states that M_c is a Lipshitz graph over a subset of $M \times \mathbb{T}$.

Let $\tilde{\Sigma}_c \subset TM \times \mathbb{T}$ be the set of all c-minimal trajectories in the phase space and let $\Sigma_c = \Pi(\tilde{\Sigma}_c)$ where $\Pi : T\mathcal{M} \times \mathbb{T} \to M \times \mathbb{T}$ is the natural projection and let $\Sigma_c^0 = \Sigma \cap M \times \{0\} \subset M \times \mathbb{T}$. For any $a, b \in M$, let

$$h_c^n(a,b) = \min_{z_0=a, z_n=b} \sum_{i=0}^{n-1} h_c(z_1, z_2, \ldots, z_n).$$

We say that a subset $S \subset \Sigma_c^0$ is *dynamically connected* if for any $a, b \in S$,

$$\liminf_{n\to\infty} h_c^n(a,b) + \liminf_{n\to\infty} h_c^n(b,a) = 0.$$

The concept of dynamically connected c-minimal trajectories will be used later when we define barrier functions.

We now return to the twist maps and we summarize some of the basic properties regarding Mather measures and c-minimal trajectories:

(1) For every irrational rotation number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, there is a unique minimal invariant measure whose support is the closure of a recurrent action-minimizing trajectory with rotation number α.

(2) For every rational number $\alpha = p/q \in \mathbb{Q}$, minimal invariant measures with rotation number p/q are supported on action-minimizing periodic orbits of period (q,p). Generically, there is only one such minimal measure for each rational rotation number.

(3) The β function is strictly convex and it is smooth at every irrational point. Let $c = \beta'(\alpha)$ for $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then the minimal measures with rotation number α is the only c-minimal measure.

(4) The β function is smooth at a rational point p/q if and only there is an invariant curve consists of periodic points of period (q,p). In this case, all these periodic orbits are $c = \beta'(p/q)$-minimal. If there is no invariant curve with periodic points of period (q,p), let c_1 be the left derivative of β at p/q and let c_2 be the right derivative of β at p/q, then the action-minimizing periodic points are c-minimal for all $c \in [c_1, c_2]$. The action-minimizing trajectories with rotation number $(p/q)^-$ are c_1-minimal and with ratation number $(p/q)^+$ are c_2-minimal. The existence of the left and right derivatives for β is due to the fact that it is convex.

(5) All bi-infinite and one-sided infinite action-minimizing trajectories are c-minimal for some c.

(6) The set of all c-minimal trajectories for a fixed c is dynamcally connected if and only if the set of all such c-minimal trajectories are connected in the phase space $T\mathbb{T} \times \mathbb{T}$. Therefore, if the c-minimal trajectories have irrational rotation numbers, then Σ_c^0 dynamically

connected. If the set of all c-minimal trajectories have rational rotation numbers, then Σ_c^0 may or may not be dynamically connected, but for the two extreme values of c for the rotation number, Σ_c^0 is dynamically connected.

6. Arnold Diffusion

We now return to the problem on $\mathbb{T}^2$ in the previous section. This time, instead of working in the covering space of $\mathbb{T}^2$, we lift the second component to $\mathbb{R}$ and preserve the first component. Recall that we need to connect trajectories from M^0 to trajectories in M^1, this could be achieved with proper choices of cohomology in the second component, but it is much more convenient to just lift it to $\mathbb{R}$. The points we consider will be in $\mathbb{T} \times \mathbb{R}$ and the trajectories will be in $\mathbb{T} \times \mathbb{R}$. Likewise, the β function and α function will be these for the Lagrangian restricted to $y = 0$ or $y = 1$.

For any $c \in \mathbb{R}$ and $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathbb{T} \times \mathbb{R}$ and let λ_c be a closed 1-form on $\mathbb{T} \times \mathbb{R}$ whose cohomology class is c, let

$$h_c(z_1, z_2) = \min \int_0^1 (L - \lambda_c)(d\gamma(t), t)dt - \alpha(c),$$

where the minimum is taken over all curves γ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. Note that x_1 and x_2 are points on $\mathbb{T}$, instead of $\mathbb{R}$. We call $h_c(z_1, z_2)$ the c-generating function. Note that the c-generating function depends on the choice of λ_c. One could just use, for example, $\lambda_c = cdx$, but appropriate choice of λ_c will be made for later convenience. With $h_c(z_1, z_2)$ we can define c-minimal trajectories as in the general case. Note that c-minimal trajectories is independent of the choices of λ_c.

We are now ready to define the barrier functions and joint barrier functions which will extend these defined for the twist maps. The basic idea is similar, except that we replace action-minimizing trajectories with c-minimizing trajectories.

Barrier functions are for connecting trajectories. There are several types of connections. We consider two types: one connects minimal trajectories from M^0 to M^1 and the other one connects minimal trajectories within M^0 or M^1. Intrinsically they are all the same. First we consider connections between M^0 and M^1. We will call the resulting barrier function P^+.

For any point $c \in \mathbb{R}$ and $a = (a_1, a_2) \in \mathbb{T} \times \mathbb{R}$ with $0 < a_2 < 1$, let $z^a \in (\mathbb{T} \times \mathbb{R})^{\mathbb{Z}}$ be a bi-infinite sequence satisfying the following properties

- $z_0^a = a$;
- $\{z_i^a\}_{i=0}^{\infty}$ is c-minimal and the ω-limit set of z^a is in M^1;
- $\{z_i^a\}_{i=-\infty}^{0}$ is c-minimal and the α-limit set of x^a is in M^0.

Such bi-infinite sequence z^a exists but may not necessarily be unique, especially when the c-minimal trajectories have rational rotation numbers. However, for any given a, the choices of z^a is in a compact set. Now we define the barrier function for such c by

$$\begin{aligned} P_c^+(a) &= \inf_{z^a} \lim_{N\to\infty} \Bigg(\Big(\sum_{i=-N}^{N-1} h_c(z_i^a, z_{i+1}^a)\Big) \\ &- \inf_{\{z \mid z_{-N}=z_{-N}^a,\, z_N=z_N^a\}} \Big(\sum_{i=-N}^{N-1} h_c(z_i, z_{i+1})\Big) \Bigg). \end{aligned}$$

An important property of the barrier function is that $P_c^+(a) \geq 0$ and $P_c^+(a) = 0$ if and only if one of the choices of z^a is c-minimal. The definition of the barrier function can in fact be further refined to reflect different structures of the c-minimal trajectories in M^0 and M^1. In essence, the barrier function can be defined with respect to dynamically connected (as defined in the previous section) closed invariant subsets instead of the c-minimal set as a whole. If c-minimal trajectories for a fixed c corresponds to action-minimizing trajectories with irrational rotation number, there is only one dynamically connected component and all z^a trajectories have the same asymptotic behavior in the phase space. If the corresponding c-minimal trajectories on M^0 has rational rotation number (q, p), then we can further divide the set of all c-minimal trajectories into disjoint union of dynamically connected subsets. We can *a priori* require that the ω-limit of z^a in one of these set and α-limit of z^a be in the same or a different subset. Generically, there is only one dynamically connected set for any c.

We can also define the barrier function for connections within M^0 or M^1.

For any point $c \in \mathbb{R}$ and $a = (a_1, a_2) \in \mathbb{T} \times \mathbb{R}$ with $-1 < a_2 < 1$, let $z^a \in (\mathbb{T} \times \mathbb{R})^{\mathbb{Z}}$ be a bi-infinite sequence satisfying the following properties

- $z_0^a = a$;
- $\{z_i^a\}_{i=0}^{\infty}$ and $\{z_i^a\}_{i=-\infty}^{0}$ are c-minimizing; Both the ω-limit set and α-limit set of z^a are contained in M^0.

Such bi-infinite sequence z^a always exists but again may not necessarily be unique. However, for any given a, the choices of z^a is in a compact set. Now we define the barrier function for such c in the same way:

$$\begin{aligned} P_c(a) &= \inf_{z^a} \lim_{N\to\infty} \Bigg(\Big(\sum_{i=-N}^{N-1} h_c(z_i^a, z_{i+1}^a)\Big) \\ &- \inf_{\{z \mid z_{-N}=z_{-N}^a,\, z_N=z_N^a\}} \Big(\sum_{i=-N}^{N-1} h_c(z_i, z_{i+1})\Big) \Bigg) \end{aligned}$$

It is important to note that for $a_2 = 0$, this new barrier function agrees with the barrier function defined for the twist map, with proper choices of rotation numbers. This is due to the fact that every action-minimizing trajectories for the twist map is c-minimal for some c.

To define constraint barrier function, let D be a closed disk contained in the strip with $\{(x, y) \in \mathbb{R}^2 | 0 < y < 1\}$, we define the barrier function with constraint D and a real number c. For any $a \in D$, let

$$\begin{aligned} P_c^+(a, D) &= \inf_{z^a} \lim_{N\to\infty} \Bigg(\Big(\sum_{i=-N}^{N-1} h_c(z_i^a, z_{i+1}^a)\Big) \\ &- \inf_{\{z \mid z_0 \in D;\ z_{-N}=z_{-N}^a,\ z_N = z_N^a\}} \Big(\sum_{i=-N}^{N-1} h_c(z_i, z_{i+1})\Big) \Bigg) \end{aligned}$$

We have the following easy proposition whose proof is straightforward.

Proposition 6.1. *Let $D \subset \mathbb{R}^2$ be a closed disk defined above and let $c \in \mathbb{R}$ be a real number. Suppose $P_c^+(b, D) > 0$ for all $b \in \partial D$, where ∂D is the boundary of D. Then there is a constraint c-minimal trajectory $z \in (\mathbb{T}\times\mathbb{R})^{\mathbb{Z}}$ with z_0 in the interior of D and z is α-asymptotic to trajectories in M_c^0 and ω-asymptotic to trajectories in M_c^1.*

Proof. Let $x^{c,0}$ be a c-minimal trajectory in M^0 and $x^{c,1}$ be a c-minimal trajectory in M^1. For any integers, $l < k$, let $z^{l,k} = \{z_i^{l,k}\}_{i=l}^k$ be a segment such that $z_l^{l,k} = x_0^{c,0}$, $z_k^{l,k} = x_0^{c,1}$, $z_0^{l,k} \in D$ and

$$h_c(z_l^{l,k}, z_{l+1}^{l,k}, \ldots, z_k^{l,k}) \leq h_c(z_l, z_{l+1}, \ldots, z_k)$$

for any segment $z = \{z_i\}_{i=l}^k$ with $z_l = z_l^{l,k}$, $z_0 \in D$ and $z_k = z_k^{l,k}$.

Let

$$B = \liminf_{l\to-\infty, k\to\infty} h_c(z_l^{l,k}, z_{l+1}^{l,k}, \ldots, z_k^{l,k}).$$

Let (l_j, k_j) be a subsequence such that $l_j \to -\infty$, $k_j \to \infty$ as $j \to \infty$ and

$$\lim_{j\to\infty} h_c(z_{l_j}^{l_j,k_j}, z_{l_j+1}^{l_j,k_j}, \ldots, z_{k_j}^{l_j,k_j}) = B.$$

The sequence of segments z^{l_j,k_j} as $j \to \infty$ has a convergent subsequence and let $z^* = \{z_i^*\}_{i\in\mathbb{Z}}$ be the limit of this subsequence. Clearly, $\{z_i^*\}_{i\geq 0}$ and $\{z_i^*\}_{i\leq 0}$ are both c-minimal and any finite segment of z^* is action-minimizing with fixed length. By the defintion of the barrier, $z^* \notin \partial D$. Therefore, z^* is a true trajectory in the phase space.

This proves the proposition. □

The set D in the above proposition is called a *barrier*.

Similarly, We define the constraint barrier function for connections within M^0. This time the constraint D is a closed disk in the strip $\{(x, y) \in \mathbb{R}^2 | -1 < y < 1\}$. We remark that for any barrier I, with length smaller than one, for the twist map defined before, any small neighborhood of I, say $I \times [-\epsilon, \epsilon]$, gives a barrier for our new barrier function $P_c(a, D)$ for some proper choice of c. This implies that all the connecting orbits with barrier we constructed for the twist map are again connecting orbits with barriers.

The idea of joint barrier function is to connect two c-minimal trajectories with two different values of c. The joint barrier is always defined with a constraint. Again, we have two different cases: connecting within M^0

or connecting between M^0 and M^1. They are similar and we will only formulate the later case.

Fix a closed simply connected set D contained in the strip with $\{(x,y) \in \mathbb{R}^2 \mid 0 < y < 1\}$ and fix two real numbers c_1 an dc_2. For a point $a = (a_1, a_2) \in D$, let z^a be a bi-infinite sequence such that

- $z_0^a = a$;
- $\{z_i^a\}_{i=0}^{\infty}$ is c_2-minimal and the ω-limit set of z^a is contained in $M_{c_2}^1$;
- $\{z_i^a\}_{i=-\infty}^{0}$ is c_1-minimal and the α-limit set of z^a is contained in $M_{c_1}^0$.

We define

$$P_{c_1,c_2}^{+}(a,D) = \inf_{z^a} \lim_{N\to\infty} \Bigg(\Big(\sum_{i=-N}^{-1} h_{c_1}(z_i^a, z_{i+1}^a) + \sum_{i=0}^{N-1} h_{c_2}(z_i^a, z_{i+1}^a) \Big)$$

$$- \inf_{\{z \mid z_0 \in D;\, z_{-N} = z_{-N}^a,\, z_N = z_N^a\}} \Big(\sum_{i=-N}^{-1} h_{c_1}(z_i, z_{i+1}) + \sum_{i=0}^{N-1} h_{c_2}(z_i, z_{i+1}) \Big) \Bigg)$$

For the above definition, we need to fix the choices of λ_{c_1} and λ_{c_2} such that the support of these two forms do not intersect D. This is always possible since D is simply connected in $\mathbb{T}\times\mathbb{R}$. We need this addition assumption so that the minimizer of

$$\sum_{i=-N}^{-1} h_{c_1}(z_i, z_{i+1}) + \sum_{i=0}^{N-1} h_{c_2}(z_i, z_{i+1})$$

with fixed end points and fixed N is actually an action-minimizing segment for $h(z_1, z_2)$ if one lift it to the covering space.

Similar to Proposition 6.1, we have the following simple proposition:

Proposition 6.2. *Let $D \subset \mathbb{R}^2$ be a closed bounded set defined above and let $c_1, c_2 \in \mathbb{R}$ be real numbers. Suppose $P_{c_1,c_2}^{+}(b,D) > 0$ for all $b \in \partial D$. Then there is a constraint action-minimizing trajectory $z \in (\mathbb{T}\times\mathbb{R})^{\mathbb{Z}}$ with z_0 in the interior of D and z is α asymptotic to trajectories in $M_{c_1}^0$ and ω asymptotic to trajectories in $M_{c_2}^1$.*

The proof of this proposition similar to that of Proposition 6.1. We ommit the details.

The set D satisfying the assumptions in the above proposition is called a *joint barrier*. The existence of a joint barrier for c_1, c_2 implies the existence of a connecting orbit between c_1-minimal and c_2-minimal trajectories.

Our next objective is to find conditions for the existence of such joint barrier. First, we consider minimal trajectories with irrational rotation numbers. Let $c \in \mathbb{R}$ be a real number such that whose corresponding c-minimal trajectories have irrational rotation number. Let z^c be a c-minimal trajectory whose α-limit is contained in M^0 and whose ω-limit is contained in M^1. We say that z_c is *weakly isolated* if there is a closed disk (simply connected) $D \subset \mathbb{T}\times(0,1)$ such that $z_0^c \in D$ and $P_c^{+}(b) > 0$, for any $b \in \partial D$, i.e., there is no c-minimal trajectory z going through the boundary of D. In fact, it is easy to show that D actually is a barrier: the constraint barrier function $P_c(b, D) > 0$ for any $b \in \partial D$. This follows from the fact that there is only one connected c-minimal set in M^0 and M^1.

By continuity of the barrier function, we have the following proposition.

Proposition 6.3. *Let $c \in \mathbb{R}$ be a real number such that all c-minimal trajectories have irrational rotation number. Let z be a weakly isolated c-minimal trajectory whose α-limit and ω-limit are contained in M^0 and M^1 respectively. Then there is a closed disk D in $\mathbb{T} \times (0,1)$ and real numbers c_- and c_+ such that for any $c_- \leq c_1, c_2 \leq c_+$, D is a joint barrier for c_1 and c_2. i.e., $P^+_{c_1,c_2}(b, D) > 0$ for all $b \in \partial D$.*

We now turn our attention to c-minimal trajectories with rational rotation numbers. Same as in the twist map case, the asymptotic behavior for such c-minimal trajectories are not unique. In this case, one can divide the recurrent c-minimal set into connected components and then treat each one of these connected components to obtain a general and detailed result. In generic situation, there is no invariant curves with rational rotation number, therefore one can always connect through rational rotations from within M^i, $i \in \mathbb{Z}$, using the extended barriers found in the twist map. For the non-generic case, we may need to connect from M^i to M^{i+1}, using barrier function P^+.

For any rational number p/q, let $[c_1, c_2]$ be the maximum interval such that the action-minimizing periodic trajectories of period (q, p) are c-minimizing for all $c \in [c_1, c_2]$. If there is an invariant curve of periodic trajectories with period (q, p), then such interval degenerates into a single point. If there is a invariant curve with rotation number p/q, then there exist $c' \in [c_1, c_2]$ such that the trajectories on this curve is c'-minimal. Thus, the set of c'-minimal trajectories on M^0 is dynamically connected and therefore Proposition 6.3 holds for such c' and one can connect through c', if there is a weakly isolated connecting orbit c'-minimal trajectory from M^0 to M^1.

For the two end points of the interval $[c_1, c_2]$, since c_1-minimal, similarly c_2-minimal, trajectories form connected sets in the phase space, with trajectories of rotation number $(p/q)^-$ connecting (q, p)-periodic trajectories. Proposition 6.3 again holds for c_1 and c_2 too.

One may also directly connect between $M^0_{c_1}$ and $M^1_{c_2}$. If there is a weakly isolated minimal connection trajectory from $M^0_{c_1}$ to $M^1_{c_2}$, then there is a joint barrier D for $P^+_{c_1,c_2}$. Again Proposition 6.3 is true for this case too.

To summarize, for any $\underline{c} < \bar{c} \in \mathbb{R}$, there exists finite numbers $c_1 = \underline{c}, c_2, \ldots, c_k = \bar{c}$ with joint barriers D_1, $D-2$, ..., D_{k-1} for the pairs (c_1, c_2), ..., (c_{k-1}, c_k) if the following conditions are satisfied: for any $c \in [\underline{c}, \bar{c}]$,

(1) if the corresponding c-minimal trajectories have irrational rotation number, then either there is a weakly isolated c-minimal connecting orbits between M^0_c and M^1_c, or there is no c-minimal invariant curve in M^0.

(2) if the corresponding c-minimal trajectories have rational rotation number, let $[\tilde{c}_1, \tilde{c}_2]$ be the maximum interval of c-values whose c-minimal trajectories have this rotation number, then either there is no invariant curves with this rational rotation number, or there is a weakly isolated minimal connecting orbits joining $M^0_{\tilde{c}_1}$ to $M^1_{\tilde{c}_2}$.

Definition 6.4. *A finite sequence of numbers c_1, c_2, c_3, ..., c_k is said to be a transition chain if there are joint barriers D_1, D_2, ..., D_{k-1} for the pairs (c_1, c_2), (c_2, c_3), ..., (c_{k-1}, c_k).*

What we summarized above gives a sufficient condition for existence of a transition chain.

Our next proposition shows the existence of (long) connecting orbits between two arbitrary points in a transition chain. The proposition is an extension of the Proposition 3.7 for the twist maps.

Proposition 6.5. *Let c_1, c_2, c_3, ..., c_k be real numbers. Let D_1, D_2, ..., D_{k-1} are joint barriers for (c_1, c_2), (c_2, c_3), ..., (c_{k-1}, c_k) either with barrier function P or P^+. Then there exists a constraint minimal trajectory z and integers $N_1 = 0 < N_2 < \ldots < N_{k-1}$ such that*

(1) *the α-limit set is contained in $M^0_{c_1}$;*
(2) *z_{N_i} is in the interior of $D_i + (0, K_{i-1})$, where K_{i-1} is the number of P^+ barrier functions before ith barrier function, recall that the barrier function is either P or P^+;*
(3) *the ω-limit set is in $M^{K_k}_{c_k}$.*

Proof. We only give a proof of the proposition with two barriers. The general case can be proved in the same way.

More precisely, let D_1 be a joint barrier for $P^+_{c_1,c_2}$ and D_2 be a joint barrier for $P^+_{c_2,c_3}$. We may assume, due to the periodicity of the system, D_2 is in the strip $S^1 \times (1,2)$, while D_1 is in the strip $S^1 \times (0,1)$. Let $\delta_1 = \inf_{b \in \partial D_1} P^+_{c_1,c_2}(b) > 0$ and $\delta_2 = \inf_{b \in \partial D_2} P^+_{c_2,c_3}(b) > 0$.

Let N_1 be a positive integer such that

$$\inf_{z^a} \Bigg(\Big(\sum_{i=-N_1}^{-1} h_{c_1}(z^a_i, z^a_{i+1}) + \sum_{i=0}^{N_1-1} h_{c_2}(z^a_i, z^a_{i+1}) \Big)$$

$$- \inf_{\{z \mid z_0 \in D;\ z_{-N} = z^a_{-N_1},\ z_N = z^a_{N_1}\}} \Big(\sum_{i=-N}^{-1} h_{c_1}(z_i, z_{i+1}) + \sum_{i=0}^{N-1} h_{c_2}(z_i, z_{i+1}) \Big) \Bigg) \geq \frac{3\delta_1}{4}$$

for all $a \in \partial D_1$. By the definition of the barrier, such N always exists. Moreover, there exists a positive number $\epsilon_1 > 0$ such that if we replace $z^a_{\pm N_1}$ by something within ϵ_1 distance in the left hand side, then the right hand side can be replaced by $\delta_1/2$ and such inequality holds for all points $a \in \partial D_1$. We may also assume that similar inequality holds for points in ∂D_2.

Fix two points $a \in D_1$ and $b \in D_2$. Let $\{z_i^n\}_{i=0}^n$ be a c_2-minimal segment such that $z_0^n = a$ and $z_n^n = b$ and

$$h_{c_2}(z_0^n, z_1^n, \dots, z_n^n) \leq h_{c_2}(z_0, z_1, \dots, z_n)$$

for any z with $z_0 = a$ and $z_n = b$. Let

$$B = \liminf_{n\to\infty} h_{c_2}(z_0^n, z_1^n, \dots, z_n^n).$$

B obviously exists and moreover it is continuous with respect to a and b.

Let $n_1, n_2, \dots,$ be a sequence of integers such that

$$\lim_{k\to\infty} h_{c_2}(z_0^{n_k}, z_1^{n_k}, \dots, z_{n_k}^{n_k}) = B$$

Let $z_0^a = a, z_1^a, z_2^a, \dots$ be a sub-sequential limit for the subsequence $z_0^{n_k}, z_1^{n_k}, \dots, z_{n_k}^{n_k}$ as $k \to \infty$. Clearly, z^a is c_2-minimal and the ω-limit set of $z_1^a, z_2^a, z_3^a, \dots$ is contained in $M_{c_2}^1$.

It follows from above that for a fixed $a \in D_1$ and $b \in D_2$ and for the given $\epsilon_1 > 0$ in the above, there exist an integer $N_2 > N_1$ and $\epsilon_2 > 0$ such that if $n \geq N_2$ and

$$|h_{c_2}(z_0^n, z_1^n, \dots, z_n^n) - B| \leq \epsilon_2$$

then there is a c_2-minimal trajectory z^a such that $|z_{N_1}^a - z_{N_1}^n| \leq \epsilon_1$. We may assume that for such N_2,

$$\inf_{n\geq N_2} h_{c_2}(z_0^n, z_1^n, \dots, z_n^n) > B - \epsilon_2.$$

Similarly, we consider the limit behavior of the segment $\{z_i^n\}_{i=0}^n$ as $n \to \infty$ from the right hand side. For this, we need to shift the sequence so that it ends at index zero. Let $w_{-n}^n, w_{1-n}^n, \dots, w_0^n = b$ be the segment such that $w_i^n = z_{i-n}^n$. The same argument shows that there exist an integer $N_2' > N_1$ and a real number $\epsilon_2' > 0$ such that if $n \geq N_2'$ and

$$|h_{c_2}(z_0^n, z_1^n, \dots, z_n^n) - B| \leq \epsilon_2'$$

then there is a c_2-minimal trajectory z^b such that $|z^b_{-N_1} - w^n_{-N_1}| \leq \epsilon_1$.

We may assume that $N_2 = N_2'$ and $\epsilon_2 = \epsilon_2'$. Moreover, by compactness of D_1 and D_2, we may choose N_2 and ϵ_2 to work all values of $a \in D_1$ and $b \in D_2$.

Let $N_3 > N_2$ be a positive integer such that for any $a \in D_1$ and $b \in D_2$,

$$|\min_{\{n\ |\ N_2\leq n\leq N_3\}} h_{c_2}(z_0^n, z_1^n, \dots, z_n^n) - B| \leq \epsilon_2.$$

Such integer N_3 clearly exists and we may choose it so that it is independent of a and b.

Let $\xi_1 \in \mathbb{T}\times\{0\}$ and $\xi_3 \in \mathbb{T}\times\{2\}$ be points such that there are respectively c_1-minimal and c_3-minimal trajectories z^{c_1} and z^{c_3} such that $z_0^{c_1} = \xi_1$ and $z_0^{c_3} = \xi_3$. For integers $k < 0$, $l > N_3$, we consider a segment $z^{k,l} = \{z_i\}_{i=k}^l$ satisfying the following properties:

(1) $z_k^{k,l} = \xi_1$ and $z_l^{k,l} = \xi_3$;
(2) $z_0^{k,l} \in D_1$ and $z_j^{k,l} \in D_2$ for some j with $N_2 \leq j \leq N_3$;

(3) $z_0^{k,l}, z_1^{k,l}, \dots z_j^{k,l}$ is (c_2, N_2, N_3)-minimal in the following sense:

$$h_c(z_0^{k,l}, \dots, z_j^{k,l}) \le h_c(z_0', \dots, z_m')$$

for any other segment $(z_0', \dots, z_m')$ such that $z_0^{k,l} = z_0'$, $z_j^{k,l} = z_m'$ and $N_2 < m < N_3$.

(4) For any segment $z = \{z\}_{i=k}^l$ satisfying above three conditions (with j replaced by an integer j'),

$$\begin{aligned} & h_{c_1}(z_k^{k,l}, \dots, z_0^{k,l}) + h_{c_2}(z_0^{k,l}, z_1^{k,l}, \dots, z_j^{k,l}) + h_{c_3}(z_j^{k,l}, \dots, z_l^{k,l}) \\ \le\ & h_{c_1}(z_k, \dots, z_0) + h_{c_2}(z_0, z_1, \dots, z_{j'}) + h_{c_3}(z_{j'}, \dots, z_l) \end{aligned}$$

For each fixed $k < 0$, $l > N_3$, such segment $\{z_i^{k,l}\}_{i=k}^l$ always exists. Let

$$\begin{aligned} B_2 = \liminf_{k \to -\infty; l \to \infty} & \Big(h_{c_1}(z_k^{k,l}, \dots, z_0^{k,l}) \\ & + h_{c_2}(z_0^{k,l}, z_1^{k,l}, \dots, z_j^{k,l}) + h_{c_3}(z_j^{k,l}, \dots, z_l^{k,l}) \Big) \end{aligned}$$

Let k_i, l_i be a subsequence such that $k_i \to -\infty$ and $l_i \to \infty$ as $i \to \infty$ and

$$\begin{aligned} \lim_{i \to \infty} \Big(h_{c_1}(z_{k_i}^{k_i,l_i}, \dots, z_0^{k_i,l_i}) + h_{c_2}(z_0^{k_i,l_i}, z_1^{k_i,l_i}, \dots, z_{j_i}^{k_i,l_i}) & \\ + h_{c_3}(z_{j_i}^{k_i,l_i}, \dots, z_{l_i}^{k_i,l_i}) \Big) & = B_2 \end{aligned}$$

Since $N_2 \le j_i \le N_3$, there is at least one j that appears infinitely many time as j_i. By choosing a subsequence, we may assume that j_i are the same for all $i \in \mathbb{N}$. Now, we take a sub-sequential limit of z^{k_i,l_i} as $i \to \infty$ to obtain a trajectory z^*. z^* satisfies the following conditions:

(1) $z_0^* \in D_1$ and $z_j^* \in D_2$ for some j with $N_2 \le j \le N_3$;
(2) $z_0^*, z_1^*, \dots z_j^*$ is (c_2, N_2, N_3)-minimal in the sense defined before;
(3) $\{z_i\}_{i \le 0}$ is c_1-minimal with limit set in M^0;
(4) $\{z_i\}_{i \ge j}$ is c_3-minimal with limit set in M^2;
(5) z^* is action-minimizing subject to the constraint $z_0^* \in D_1$ and $z_j^* \in D_2$ for some j with $N_2 \le j \le N_3$.

By the choice of N_2 and N_3, together the above properties, we see that $z_0^* \notin \partial D_1$ and $z_j^* \notin \partial D_2$. Therefore, z^* is a true trajectory in the phase space.

This proves the proposition. □

We summarize the results we obtained in this section in the following theorem.

Theorem 6.6. *Let $[\underline{c}, \bar{c}]$ be an interval in the real line such that for any $c \in [\underline{c}, \bar{c}]$, either there is no c-minimal invariant curve in M^0 or there is a weakly isolated c-minimal connecting orbit from M_c^0 to M_c^1, then for any $c_1, c_2 \in [\underline{c}, \bar{c}]$, there is a constraint action-minimizing trajectory in the phase space whose α-limit set is contained in M_{c_1} and whose ω-limit set is contained in M_{c_2}.*

Furthermore, for any Lagrangian system L satisfying the conditions we listed and for any real numbers $\underline{c} < \bar{c}$, there is a $\epsilon > 0$ and an open and dense subset V of Lagrangians in the set of all Lagrangians ϵ-close to L, such that for any $c_1, c_2 \in [\underline{c}, \bar{c}]$, there is a transition chain from c_1 to c_2 and hence there is an action-minimizing trajectory connecting M_{c_1} to M_{c_2}.

The second part of the theorem follows from the fact that for any c, the c-minimal connecting orbit being weakly isolated is a open and dense property for the Lagrangian.

7. Local variational principle

In this section, we consider the hamiltonian flow $\phi_F(t, \dot{)}$ on $\mathbb{R}^2$ generated by a smooth Hamiltonian F on $\mathbb{R}^2$. We assume that ϕ_F has hyperbolic restpoint at, say, the origin and there is a homoclinic loop connecting the restpoint back to itself, i.e., there is point $q \in \mathbb{R}^2$ such that $\lim_{t\to\pm\infty} \phi_F(t, q) = (0, 0)$. Since the Hamiltonian F is autonomous, the homoclinic loop can be described by $\Gamma = \{\phi_F(t, q) \mid t \in \mathbb{R}\}$.

We will restrict ourselves to a neighborhood of the homoclinic loop. The goal of this section is to embed it into a cylinder with the so-called positive twist condition so that we can apply the results of the previous sections. In a small neigborhood of a hyperbolic restpoint, this embdedding can be easily accomplished:

Proposition 7.1. *Let the origin be a hyperbolic respoint of a Hamiltonian flow on $\mathbb{R}^2$, there is a local coordinate in a neighborhood of the origin such that the Hamiltonian function is given by $F(\xi, \eta) = \frac{1}{2}\xi^2 - \frac{1}{2}k_1\eta^2$ for some positive real number $k_1 > 0$.*

Proof. The normal form for a hyperbolic restpoint is $F(\xi, \eta) = \frac{1}{2}\lambda(\eta^2 - \xi^2)$ for some real number $\lambda \neq 0$. This normal form actually converges. We may assume $\lambda > 0$ by rotating the coordinate by $\pi/2$ if neccessary. One more change of coordinate: $\xi \to \sqrt{\lambda}\xi$, $\eta \to \eta/\sqrt{\lambda}$ proves the proposition. $\square$

The corresponding Lagrangian for the Hamiltonian flow nearby the origin is given by $L = \frac{1}{2}\dot{\xi} + \frac{1}{2}k_1\xi^2$. Obviously, it is positive definite.

To obtain positive definiteness along a homoclinic loop, we pick a finite segment of a regular trajectory. The flow box theorem states that there are corrdinates (ξ, η) such that the Hamiltonian takes the form $F(\xi, \eta) = \eta$ and the flow lines are parallel to the ξ-axis with uniform speed. For any function $c(\eta)$ with $c'(\eta) \neq -1$, a symplectic change of coordinate by $\eta \to \eta + c(\eta), \xi \to \xi/(1 + c'(\eta))$ transforms the Hamiltonian to $F(\xi, \eta) = \eta + c(\eta)$, which is positive definite provided $c''(\eta) > 0$. The flow for this new Hamiltonian will be our basic flow box and it can be deccribed as:

$$B = \{(\xi, \eta) \mid -\delta \leq \eta \leq \delta, 0 \leq \xi \leq (1 + c'(\eta))\}.$$

A neccessary condition for embedding such a flow box is that the right hand side of the box positively slanted.

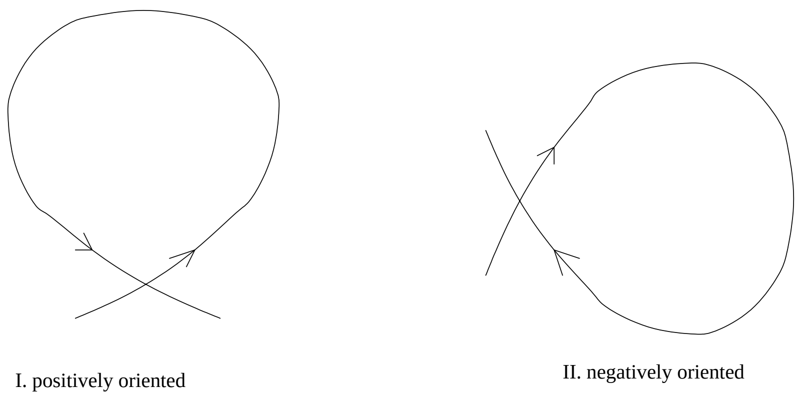

FIGURE 1. orientations of homoclinic loops

Next we observe that there are two types of orientations of the homoclinic loop, as illustrated in Figure 7. We call the type I positively oriented and the type II negatively oriented.

For the positively oriented loop, we can always embed a small neighborhood of the loop into a cylinder $\mathbb{T} \times \mathbb{R}$ such that the loop is a graph over the circle $\mathbb{T} \times \{0\}$. Following a thorem of Moser [7], this embedding can be made area-preserving. Let $(\xi(\mod 1), \eta)$ be a coordinate on $\mathbb{T} \times \mathbb{R}$, we can further require that the Hamiltonian on the cylinder takes the form $F(\xi, \eta) = \frac{1}{2}\eta^2 + \frac{1}{2}k_1\xi^2$ near $\xi = \eta = 0$ and the loop is above the circle $\mathbb{T} \times \{0\}$.

Let l be a small vertical line cross the homoclinic loop and this line is contained in a neighborhood of $\xi = \eta = 0$. We follow this vertical line along the flow and shrink the length of l if neccessary. Eventually, the image of this line under the flow, $\phi_t(l)$, is close to the point $(1, 0)$ for large $t > 0$. See figure 7. Following the λ Lemma, the image of the line, $\phi_T(l)$ will be c^1-close to the unstable manifold of $(1, 0)$ for sufficiently large T and therefore, it will be postively slanted if l is chosen small enough. Now we replace the flow between the line l and $\phi_T(l)$ with a new one, similar to the flow box we introduced before. Disregard the area-preserving requirement, we parametrize the new flow in the box, ψ_t, $0 \le t \le T$, such that (1). the original Hamiltonian flow, ϕ_t, join smoothly with ψ_t near l and $\phi_T(l)$; (2). let (v_ξ, v_η) be the vector field generated by ψ_t, then $\partial v_\xi / \partial \eta > 0$, for all (ξ, η) in the box. This is possible since the right hand side of the box is positively slanted. We call this flow box B.

The flow in this new flow box defined above is not neccessarily area-preserving with respect to the standard area form. However, it is area-preserving with a different area-form, σ, induced by the flow ψ_t. The area-form σ agrees with the standard area-form at left and right sides of the box B. Finally, we take a new embedding of B onto a subset B' of cylinder such that (1). the embedding is area-preserving with respect to the area-form σ

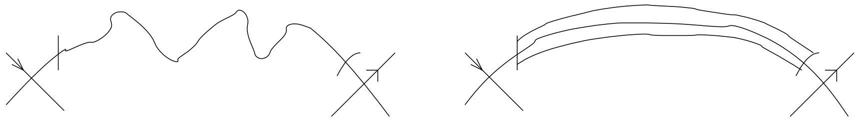

FIGURE 2. inserting positive definite flow box

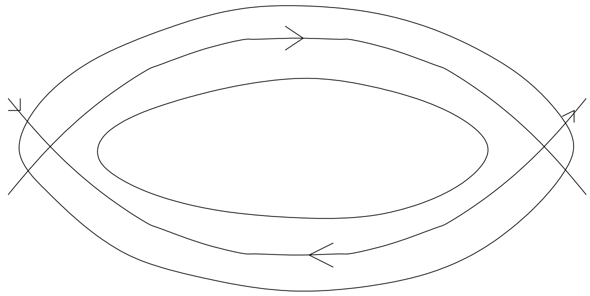

FIGURE 3. embedding negatively oriented loop

for B and the standard area-form for B'; (2). the embedding preserves the vertical lines, and (3). the embedding preserves the left and right sides of the box B. It is easy to see that such embedding exists and if, in addition, one fixes the homoclinic loop, then such embedding is unique. Under the new coordinate, the flow is area preserving and hence Hamiltonian with a positive definite Hamiltonian in a small neighborhood of the homoclinic loop.

To summarize, for the positively oriented case, there is a Hamiltonian function F defined on a neighborhood of the invariant homoclinic loop such that F satisfies the positive definite condition, i.e., $F_{\eta\eta}(\xi, \eta) > 0$.

For the negatively oriented homoclinic loop, we first take a double cover of its small neighborhood. We obtain two positive definite restpoints, at $(0, 0)$ and $(1, 0)$, with two heteroclinic connections. Similar to the positively oriented case, this can be embedded in $\mathbb{R}^2$ such that the Hamiltonian near these loops are positive definite. See Figure 7.

To simplify the notations, from now on, we will focus our attention to the positively oriented case. We will point out the neccessary changes for the negatively oriented case.

From the locally defined Hamiltonian functions, the Legendre transformation gives locally defined Lagrangian. For the positvely oriented case, the Lagrangian takes the following form: $L = L(\dot{\xi}, \xi)$ for all $(\xi, \dot{\xi}) \in U$ where U is a strip in the cylinder containing the point $(0, 0)$. In a small neigborhood of $(0, 0)$, $L = \frac{1}{2}\dot{\xi}^2 + \frac{1}{2}k_1\xi^2$.

With L defined above, we can define a variational principle in a local sense and from there, we can define action-minimizing trajectories: a curve

$\gamma : [a, b] \to \mathbb{T}$ is action-minimizing, if $(\gamma(t), \gamma'(t)) \in U$ for all $t \in [a, b]$ and

$$\int_{t=a}^{b} L(d\gamma(t), t)dt \le \int_{t=a}^{b} L(d\tilde{\gamma}(t), t)dt$$

for any $\tilde{\gamma}$ homologous to γ relative to the end points and $(\tilde{\gamma}(t), \tilde{\gamma}'(t)) \in U$ for all $t \in [a, b]$. A smooth curve $\gamma(t)$ such that $(\gamma(t), \gamma'(t)) \in U$ is called U-constraint. A discrete trajectory $\ldots, x_i, \ldots$ is said to be U constraint if there exists a U constraint curve $\gamma(t)$ such that $\gamma(i) = x_i$.

Action-miniming trajectories in the interior of U are true trajectories and clearly the trajectories on homoclinic loop are obviously action-minimizing.

Now we consider our 1-periodic Hamitonian

$$H_\epsilon(\xi, \eta, I, \theta, t) = F(\xi, \eta) + G(I) + H^1(\xi, \eta, I, \theta, t, \epsilon),$$

where $\xi \in \mathbb{T}, \eta \in \mathbb{R}$, $I \in \mathbb{R}$ and $\theta \in \mathbb{T}$ and ϵ is a small parameter such that $H^1 = 0$ if $\epsilon = 0$. We assume that $G''(I) > 0$ for all $I \in \mathbb{R}$ and $F(\xi, \eta)$ is positive definite around a homoclinic loop as described above. The Hamiltonian flow is defined in an 5-dimensional manifold with the Poincaré map on a subset of 4-dimensional manifold.

For $\epsilon = 0$, we have two decoupled systems with Hamiltonians given by $F(\xi, \eta)$ and $G(I)$. The set defined by

$$N_0 = \{(\xi, \eta, I, \theta, t) \mid \xi = \eta = 0\}$$

is a normally hyperbolic invariant set. This invariant set N is connected to itself by a homoclinic loop. For ϵ small, by the standard invariant manifold theory, this normally hyperbolic invariant set N_0 persists if one restricts it to a bounded region. We let this perturbed manifold be N_ϵ and N_ϵ is undertood to be in a bounded region only. N_ϵ can be written as a graph over the original unperturbed manifold N_0. We may assume, by a symplectic change of coordinate, that N_ϵ is still defined by $\xi = \eta = 0$.

We now take a Legender transformation to the Hamiltonian system, replacing ξ and θ by x and y to confirm with our notations of the previous sections, and we obtain a Lagrangian function L on $\mathbb{T}^2$. $L = L(x, y, \dot{x}, \dot{y}, t)$ satisfies the following properties:

(1) L is defined on $U \times \mathbb{T} \times \mathbb{R} \times \mathbb{T}$, where U is a strip on the cylinder containing the point $(0, 0)$ in our chosen cylinderical coordinate. Note that L depends also on ϵ, which is not explicitly expressed in our notation.

(2) L is positive definite on its domain of definition.

(3) For any (large) real number $K > 0$, there exist real numbers $\epsilon_0 > 0$, $\delta > 0$, $\sigma > 0$ and integer $N > 0$ such that the following is true:

Fix $z_0 = (x_0, y_0), z_T = (x_T, y_T) \in (-\delta, 1 + \delta) \times \mathbb{R}$, $|y_T - y_0| < KT$, let $\gamma : [0, T] \to \mathbb{R}^2$ be a U-constraint action-minimizing curve with $\gamma(0) = z_0$ and $\gamma(T) = z_T$, $T > N$, for the Lagrangian with $|\epsilon| \le \epsilon_0$. If $|x_0| \le \delta$ or $|1 - x_T| \le \delta$, then such γ exist and it is in the interior of

U. Moreover, for any U-constraint curve $\tilde{\gamma}$, homologous to γ relative to the endpoints, if $\tilde{\gamma}$ is on the boundary of U at any point, then

$$\int_{t=a}^{T} L(d\gamma(t), t)dt \leq \int_{t=a}^{T} L(d\tilde{\gamma}(t), t)dt - \sigma$$

To prove the third property in the above, we note that it is true for $\epsilon = 0$, for δ sufficiently small and T sufficiently large. The action-minimizing trajectories stay close to the unstable manifold of $(0, 0)$, or the stable manifold of $(1, 0)$. For each choice of z_0 and z_T, the property is true for small ϵ. Restricting to $|y_T - y_0| < BT$, we have a uniform bound ϵ_0 for ϵ.

Now we use techniques introduced in the previous sections to prove our main theorem. We need to change all free configurations to U-constraint configurations and we need to restrict ourselves to a bounded subset, which implies we need to be restricted to bounded rotation numbers. This bound can be arbitrarily large as long as we choose ϵ small. Finally, we need to verify all the conditions $(H_a - H_c)$, whic folows easily from above properties.

For the negatively oriented case, instead of constructing connecting orbits from M^i to M^{i+1} for integer i, we construct connecting orbits from M^0 to M^1 and then from M^1 back to M^0. In each connection with put a barrier around M^0 or M^1 so that trajectories do not stay in the same invariant set. The rest of the construction is the same.

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