

COUNTEREXAMPLE TO CONJECTURED $SU(N)$ CHARACTER ASYMPTOTICS

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1. INTRODUCTION

The purpose of this note is to give a counterexample to the conjectured large N asymptotics of character values $\chi_R(U)$ of irreducible characters of $SU(N)$, which appears in papers of Gross-Matytsin [M, GM] and Kazakov-Wynter [KW]. Asymptotics of characters are important in the large N limit in YM_2 (2D Yang-Mills theory) and in certain matrix models [KSW, KSW2, KSW3]). Our counterexample consists of one special sequence of elements $a_N \in SU(N)$ for which the conjectured asymptotics on $\chi_{R_N}(a_N)$ fail for any relevant sequence χ_{R_N} of irreducible characters. It is not clear at present how widespread in $SU(N)$ the failure is.

To state the conjecture and the counterexample, we will need some notation. We recall that irreducibles of $SU(N)$ are parametrized by their highest weights λ or equivalently by Young diagrams with $\leq N - 1$ rows. To facilitate comparison with [GM], we will use a further parametrization of representations R of $SU(N)$ by their shifted highest weights

$$\ell = \lambda + \rho_N, \quad \rho_N = \text{half the sum of the positive roots.} \quad (1)$$

The components of the shifted highest weight are then strictly decreasing $\infty > \ell_1 > \ell_2 > \dots > \ell_N > -\infty$ (cf. (13) for the explicit formula). To a shifted highest weight we associate the probability measure on \mathbb{R} defined by

$$d\rho_R = \frac{1}{N} \sum_{j=1}^N \delta\left(\frac{\ell_j}{N}\right), \quad \text{i.e.} \quad \int_{\mathbb{R}} f(y) d\rho_R(y) = \frac{1}{N} \sum_{j=1}^N f\left(\frac{\ell_j}{N}\right). \quad (2)$$

Given a sequence R_N of irreducible representations of $SU(N)$, we write

$$R_N \rightarrow d\rho, \quad \text{if} \quad d\rho_{R_N} \rightarrow d\rho \quad \text{in the sense of measures.} \quad (3)$$

Any weak limit is a probability measure satisfying $\rho_Y([0, T]) \leq T$, since $\ell_j - \ell_{j+1} \geq 1$. If the limit has a density, which is written $d\rho_Y = \rho'_Y(y)dy$, then $\rho'_Y(y) \leq 1$. A limit measure is called a “distribution on Young tableaux”.

The conjecture of Gross-Matytsin, Kazakov-Wynter and other physicists concerns the values of a sequence of characters χ_{R_N} on elements U_N of $SU(N)$. The eigenvalue distribution

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of $U \in SU(N)$ with eigenvalues $\{e^{i\theta_k}\}$ is the probability measure on the unit circle S^1 defined

$$d\sigma_N := \frac{1}{N} \sum_{k=1}^N \delta(e^{i\theta_k}).$$

Given a sequence $U_N \in SU(N)$, we write $U_N \rightarrow \sigma$ (as $N \rightarrow \infty$) if $d\sigma_N \rightarrow \sigma$ in the sense of measures, i.e. $\frac{1}{N} \sum_{k=1}^N f(e^{i\theta_k}) \rightarrow \int_{S^1} f d\sigma$.

CONJECTURE 1.1. (*Gross-Matytsin [GM], (2.3); Kazakov-Wynter [KW], Appendix 5.1; [KSW], §3; see below*) Assume $U_N \rightarrow \sigma, R_N \rightarrow \rho$. Then $\chi_{R_N}(U_N) \sim e^{N^2 F_0[\rho, \sigma]}$ where

$$F_0(\rho, \sigma) = S(\rho, \sigma) + \frac{1}{2} \left\{ \int_{\mathbb{R}} \rho(x) x^2 dx + \int \sigma(y) y^2 dy \right. \\ \left. - \frac{1}{2} \left\{ \int_{\mathbb{R} \times \mathbb{R}} \rho(x) \rho(y) \ln |x - y| dx dy + \int_{\mathbb{R} \times \mathbb{R}} \sigma(x) \sigma(y) \ln |x - y| dx dy \right\} \right\},$$

where S is the classical action corresponding to the Hopf equation

$$\begin{cases} \frac{\partial f}{\partial t} + f \frac{\partial f}{\partial x} = 0 \\ \Im f(x, 0) = \pi \rho(x), \quad \Im f(x, 1) = \pi \sigma(x). \end{cases}$$

Our counterexample is based on the special sequence $U_N = a_N$ of *principal elements of type ρ* of $SU(N)$ in the sense of Kostant [Ko]. Such a principal element is regular and has minimal order in $SU(N)$, given by its Coxeter number N . The eigenvalues of a_N are thus the distinct N th roots of unity, and the limit distribution $d\sigma$ of a_N of eigenvalues is obviously $d\theta$.

The key fact, discovered by Kostant [Ko] (see also [Ko2, AF]) is that characters take on only the three values

$$\chi_R(a_N) = 0, \pm 1, \quad \forall R \in \widehat{U(N)}. \quad (4)$$

This immediately casts doubt on the conjecture, since it would imply:

$$e^{N^2 F_0[\rho, d\theta]} \sim \chi_R(a_N) = \begin{cases} (i) & 0 \\ (ii) & -1, \quad \forall \rho. \\ (iii) & 1 \end{cases} \quad (5)$$

Clearly, this would require that, for all $d\rho$,

$$F_0[\rho, d\theta] \text{ is } \begin{cases} (i) & < 0 \\ (ii) & = i\pi(2k_N + 1) \\ (iii) & = o(1/N^2) \end{cases} \quad (6)$$

The following result shows that the oscillation of values of $\chi_R(a_N)$ is much too regular for any such results. There is simply no separation of the possible limiting shapes of Young diagrams into the three discrete classes of possible limits $0, \pm 1$; all possible limit shapes are consistent with the limit 0.

THEOREM 1.2. *Given any sequence of irreducibles $R_N \in \widehat{SU(N)}$, with $R_N \rightarrow \rho$, there exists a sequence $R'_N \in \widehat{SU(N)}$ with $R'_N \rightarrow \rho$ with the property that $\chi_{R'_N}(a_N) = 0$. Hence, there cannot exist a limit functional $F_0(d\theta, d\rho)$ depending only on the limit densities $d\sigma, d\rho$.*

The basic idea of the proof is the following: suppose that the highest weight R is such that $\chi_R(a_N) = \pm 1$. Then, by changing one component of R by one unit, one obtains a highest weight R' such that $\chi_{R'}(a_N) = 0$. Taking a sequence $R_N \rightarrow \rho$ and changing $R_N \rightarrow R'_N$ one obtains a new sequence with $R'_N \rightarrow \rho$ and with $\chi_{R'_N}(a_N) \equiv 0$.

1.1. Background of the conjecture. The Conjecture 1.1 attributed above to Gross-Matytsin and Kazakov-Wynter seems to have appeared independently in the papers [GM] and [KW]. It is analogous to and inspired by Matytsin's conjecture [M] on the large N asymptotics of Itzykson-Zuber integrals. The latter conjecture has recently been proved by Guionnet-Zeitouni [GZ] and Guionnet [G]. But the former is incorrect in general. We now explain the difference between the two conjectures and give some background on the context in which the conjecture arose.

The original conjecture of Matytsin pertained to integrals known variously as Itzykson-Zuber or spherical integrals

$$I(A, B) \equiv \int_{SU(N)} e^{N \text{tr}[AUBU^\dagger]} dU, \quad (7)$$

where A and B are $N \times N$ Hermitian matrices and dU is (unit mass) Haar measure on $SU(N)$. By the Itzykson – Zuber (Harish-Chandra) formula one has

$$I(A, B) = \frac{\det[e^{Na_i b_j}]}{\Delta(a)\Delta(b)}, \quad (8)$$

where $\{a_i\}$, resp. $\{b_j\}$, are the eigenvalues of A , resp. B and where $\Delta(a)$ denotes the Van der Monde determinant $\Delta(a) = \prod_{i < j} (a_i - a_j)$. In [M], Matytsin stated Conjecture 1.1 precisely in the same form for $I_N(A_N, B_N)$. This conjecture has recently been proved by Guionnet-Zeitouni [GZ] and Guionnet [G].

In the subsequent papers [GM, KW], Gross-Matytsin and Kazakov-Wynter stated an analogous conjecture for characters of $U(N)$. We quote their statements in some detail to draw attention to the key difference to Matytsin's original conjecture.

First, we consider [GM]. There are some slight differences in notation (e.g. their Ξ is our F_0) which we leave to the reader to adjust. They write: “for large N the $U(N)$ characters behave asymptotically as

$$\chi_R(U) \simeq e^{N^2 \Xi[\rho_Y(l/N), \sigma(\theta)]} \quad (9)$$

with some finite functional $\Xi[\rho_Y, \sigma]$. In this formula it is implicit that we take the limit $N \rightarrow \infty$ assuming that the eigenvalue distribution of the unitary $N \times N$ matrix U converges to a smooth function $\sigma(\theta)$, $\theta \in [0, 2\pi]$. (The eigenvalues of a unitary matrix lie on the unit circle in the complex plane and can be parametrized as $\lambda_j = e^{i\theta_j}$.) In addition, it is assumed that the distribution of parameters $\tilde{y}_i = l_i/N$, which define the representation R , also converges to another smooth function $\rho_Y(\tilde{y})$, that we can call the Young tableau density. The functional Ξ is, in general, not easy to calculate. However, in some important cases it can be found explicitly.” They continue: “...we will have to evaluate the functional derivatives of $\Xi[\rho_Y, \sigma_1]$... This can be done if we observe that the $U(N)$ characters can be

represented as analytic continuations of the Itzykson–Zuber integral (7). Setting $a_k = l_k$, $b_j = \theta_j$ and analytically continuing $a_k \rightarrow ia_k$, we see that

$$\frac{\det[e^{Na_i b_j}]}{\Delta(a)\Delta(b)} \rightarrow J(e^{i\theta_s}) \chi_R(U). \quad (10)$$

Therefore, we can use the known expressions for the large N limit of the Itzykson–Zuber integral to find the functional Ξ . In particular, if as $N \rightarrow \infty$ the distributions of $\{a_k\}$ and $\{b_j\}$ converge to smooth functions $\alpha(a)$ and $\beta(b)$, then asymptotically...” the formula in Theorem 1.2 holds with $\alpha da = \rho$, $\beta db = \sigma$.

In [KW], it is pointed out that character values for $U(N)$ are, ‘up to a factor of i ...the Itzykson-Zuber determinant...From Matysin’s paper we quote the result (with the minor change of an extra factor of i)...’

We note that the relation (10) between Itzykson-Zuber integrals and characters is the Kirillov character formula, see e.g. Theorem 8.4 of [BGV]. Thus, it is precisely the analytic continuation of the large N asymptotics of the Itzykson -Zuber integral from Hermitian to skew-Hermitian matrices (the Lie algebra of $U(N)$), i.e. the extra factor of i , which leads in general to incorrect results. The same error then propagates to the conjecture of Gross-Matytsin and Kazakov-Wynter on the large N asymptotics of the partition function of 2D $SU(N)$ Yang-Mills theory on a cylinder, which the second author disproved by a related counterexample in [Z]. On the positive side, the proof of Guionnet-Zeitouni of Matysin’s conjecture suggests that the conjectured partition function asymptotics might be correct after analytically continuing the partition function from $U(N)$ to positive matrices.

In the study of matrix models, Kazakov-Staudacher-Wynter [KSW, KSW2, KSW3] employ related asymptotics for matrices satisfying certain moment conditions (i.e. on traces of powers). These do not appear to exclude unitary matrices. It is not clear if they exclude the counterexamples we are presenting.

Of course, the counterexample does not indicate the limit of validity of the original conjectures or of their applications in 2D gravity, YM_2 and matrix models. V. Kazakov has raised a number of interesting questions regarding the counterexample. Can one perturb the counterexample or does it depend on the eigenvalues being roots of unity? Are the conjectures even ‘generically correct’ in a reasonable sense? Rather than studying pointwise limits, one can study asymptotics of statistical aspects of character values. The large deviations theory of Guionnet-Zeitouni [GZ, G] does not seem to adapt in a straightforward way to character values on $SU(N)$. What is a good probabilistic framework? Some interesting work in the statistical direction is found in the works of Kazakov-Staudacher-Wynter (loc. cit.). M. R. Douglas has suggested a different point of view, connecting large N limits with conformal field theory [D1, D2].

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2. REVIEW OF THE KOSTANT IDENTITY

In this section, we shall review the Kostant identity (4) for the values $\chi_R(a_N)$ of irreducible characters at the principal elements of type ρ . We also review a version of the formula obtained in [AF] for the group $SU(N)$.

2.1. The Kostant identity in general. Let G be a compact, connected, simply-connected semisimple Lie group. We also assume that G is simply-laced, that is, each root has the same length with respect to the Killing inner product. Let a_ρ denote a principal element of type ρ . The element a_ρ is in the conjugacy class of the element $\exp(\kappa^{-1}(2\rho))$, where ρ is half the sum of the positive roots and κ is the isomorphism between the Lie algebra of a fixed maximal torus and its dual induced by the Killing form. Since the characters are class functions, we can set $a_\rho = \exp(\kappa^{-1}(2\rho))$.

Let Λ^* denote the root lattice. Let h be the Coxeter number. The number h is defined as the order of the Coxeter element in the Weyl group W , namely the element $s_{\alpha_1} \cdots s_{\alpha_l}$, where $\{\alpha_j\}$ are the simple roots, $s_{\alpha_j} \in W$ is the reflection corresponding to α_j and l is the rank of G . The following lemma (Lemma 3.5.2 in [Ko]) is one of the key points of [Ko].

LEMMA 2.1. *Let λ be a dominant weight. Then, either*

- (1) *For all $w \in W$, $w(\lambda + \rho) - \rho \notin h\Lambda^*$ or*
- (2) *There exists a unique $w \in W$ such that $w(\lambda + \rho) - \rho \in h\Lambda^*$.*

It should be noted that, if λ satisfies the condition (2) in Lemma 2.1, then $\lambda \in \Lambda^*$, since $w\rho - \rho$ is in the root lattice Λ^* for all $w \in W$, and the lattice $h\Lambda^*$ is invariant under W -action.

By using Lemma 2.1, we define $\varepsilon(\lambda) \in \{0, \pm 1\}$ for each dominant weight λ as follows:

$$\varepsilon(\lambda) = \begin{cases} \operatorname{sgn}(w) & \text{if } \lambda \text{ satisfies (2) in Lemma 2.1,} \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

Then, the Kostant identity can be stated as follows:

THEOREM 2.2 (Kostant[Ko]). *Under the assumption on G stated above, the irreducible characters χ_λ take one of the values 0, 1 or -1 at the element a_ρ . More precisely, one has*

$$\chi_\lambda(a_\rho) = \varepsilon(\lambda)$$

for each dominant weight λ .

2.2. The Kostant identity for $SU(N)$. Now we set $G = SU(N)$. In this case, a dominant weight is regarded as a partition of a non-negative integer, and one can rewrite Theorem 2.2 in terms of a property of components of partitions. We refer the readers to [FH] for a general theory of the representation theory of $SU(N)$ and partitions, and to [AF] for a version of Kostant's theorem (Theorem 2.2) for $SU(N)$, which we shall review in this section.

To fix notation, we first define a correspondence between the dominant weights and partitions. Let \mathfrak{t}_N and \mathfrak{h}_N denote the Lie algebras of maximal tori in $SU(N)$ and $U(N)$ respectively, and let L_N^* and I_N^* denote the weight lattices in the dual spaces \mathfrak{t}_N^* and \mathfrak{h}_N^* respectively. We denote the standard basis in \mathfrak{h}_N^* by e_j , $j = 1, \dots, N$. Then, for each $\mu \in L_N^*$, there is a unique $f = f_\mu \in I_N^*$ such that

$$f = \sum_{j=1}^{N-1} f_j e_j, \quad f|_{\mathfrak{t}_N} = \mu.$$

Therefore, the weight lattice L_N^* is identified with the sublattice (of rank $N - 1$) in I_N^* spanned by e_1, \dots, e_{N-1} :

$$L_N^* \cong \bigoplus_{j=1}^{N-1} \mathbb{Z} \cdot e_j \subset I_N^*. \quad (12)$$

The roots for $(\mathfrak{su}(N), \mathfrak{t}_N)$ are given by the restrictions to \mathfrak{t}_N of the following elements in I_N^* (which are the roots for $(\mathfrak{u}(N), \mathfrak{h}_N)$): $\alpha_{i,j} = e_i - e_j$, $1 \leq i \neq j \leq N$. We take the positive roots to be $\alpha_{i,j}$ ($i < j$), and the simple roots to be $\alpha_j := \alpha_{j,j+1}$, $j = 1, \dots, N - 1$. The corresponding positive open Weyl chamber C is given, in terms of the identification (12), by

$$C = \{f = \sum_{j=1}^{N-1} f_j e_j \in \mathfrak{t}_N^*; f_1 > \dots > f_{N-1} > 0\}.$$

Thus, the set of dominant weights $P_N := \overline{C} \cap L_N^*$ is given by

$$P_N = \{f = \sum_{j=1}^{N-1} f_j e_j; f_1 \geq \dots \geq f_{N-1} \geq 0, \quad f_j \in \mathbb{Z}\},$$

which is the set of partitions of length N whose last component is zero. In this notation, half the sum of the positive roots ρ_N is given by

$$\rho_N = \sum_{j=1}^{N-1} (N - j) e_j. \quad (13)$$

The principal element of type ρ , $a_N := \exp(\kappa^{-1}(2\rho_N))$, is given by

$$a_N = \text{diag}(e^{\pi i(N-1)/N}, e^{\pi i(N-3)/N}, \dots, e^{-\pi i(N-3)/N}, e^{-\pi i(N-1)/N}), \quad (14)$$

and, in particular, the distribution of the eigenvalues of a_N tends to the normalized Haar measure on the circle.

The Kostant identity (Theorem 2.2) can be rewritten in the following form, which is obtained in [AF]:

PROPOSITION 2.3 ([AF]). *Let λ be a dominant weight for $SU(N)$, and let χ_λ be the irreducible character for $SU(N)$ corresponding to λ . As above, we write $\lambda = (\lambda_1, \dots, \lambda_{N-1}, \lambda_N)$ with $\lambda_N = 0$. Let $a_N = \exp(\kappa^{-1}(2\rho_N))$. Then, $\chi_\lambda(a_N) \neq 0$ if and only if $\lambda_j + N - j$'s have distinct residue modulo N . In such a case, we have*

$$\chi_\lambda(a_N) = \text{sgn}(\sigma),$$

where $\sigma \in \mathfrak{S}_N$ is defined by

$$\sigma(j) = N - r(j), \quad j = 1, \dots, N,$$

and $r(j)$ denotes the residue of $\lambda_j + N - j - \ell$ modulo N with $\ell = |\lambda|/N$.

Note that if $\lambda_j + N - j$'s have distinct residue modulo N , then $|\lambda|$ is automatically a multiple of N . In Proposition 2.3, the numbers $\lambda_j + N - j$ are the components of the dominant weight $\lambda + \rho_N$:

$$\lambda + \rho_N = \sum_{j=1}^{N-1} (\lambda_j + N - j) e_j.$$

The shifted highest weight $\lambda + \rho_N$ is written as ℓ in (1).

3. PROOF OF PROPOSITION 2.3 AND THEOREM 1.2

We prepare for the proofs with a series of Lemmas. For $SU(N)$, it is easy to see that the Coxeter number h is equal to N . This is proved, for example, by showing that the Coxeter element is just a cycle of length N .

The following condition specializes condition (2) in Lemma 2.1 to $SU(N)$:

$$(K) \quad \text{there exists a unique } w \in \mathfrak{S}_N \text{ such that } w(\lambda + \rho_N) - \rho_N \in N\Lambda^*.$$

LEMMA 3.1. *Let $\mu \in L_N^*$. Then $\mu \in \Lambda^*$ if and only if $|f_\mu| \in N\mathbb{Z}$, where $|f_\mu| = \sum_{j=1}^{N-1} f_j$, $f_\mu = \sum_{j=1}^{N-1} f_j e_j \in I_N^*$, $f_\mu|_{\mathfrak{t}_N} = \mu$.*

Proof. We set $e_0 = \sum_{j=1}^N e_j$ which is a weight for $\mathfrak{u}(N)$, and also set $H_0 = \sum_{j=1}^N H_j$, where H_j is the standard basis for the Lie algebra \mathfrak{h}_N of the maximal torus in $U(N)$. Then, we have $\mathfrak{t}_N^* = \mathfrak{h}_N^*/\mathbb{R}e_0$. We first claim that

$$\Lambda^* = \{f \in I_N^*; f(H_0) = 0\}/\mathbb{Z}e_0. \quad (15)$$

To prove (15), we recall that the root lattice Λ^* is spanned by the simple roots:

$$\Lambda^* = \bigoplus_{j=1}^{N-1} \mathbb{Z}\alpha_j, \quad \alpha_j = e_j - e_{j+1}. \quad (16)$$

Thus, any $\mu \in \Lambda^*$ is expressed as $\mu = \sum_{j=1}^{N-1} c_j \alpha_j$, $c_j \in \mathbb{Z}$. We define $f_\mu \in I_N^*$ by

$$f_\mu = c_1 e_1 + \sum_{j=2}^{N-1} (c_j - c_{j-1}) e_j + c_{N-1} e_N.$$

Then, clearly we have $f_\mu(H_0) = 0$ and $f_\mu|_{\mathfrak{t}_N} = \mu$, which shows (15).

Now, let $\mu \in L_N^*$. As before, we identify μ with a weight $f = f_\mu \in I_N^*$ of the form:

$$f = \sum_{j=1}^{N-1} f_j e_j, \quad f_j \in \mathbb{Z}, \quad j = 1, \dots, N-1, \quad f|_{\mathfrak{t}_N} = \mu.$$

In the above, we sometimes set $f_N = 0$.

First, assume that $|f| = f(H_0) \in N\mathbb{Z}$. We define a weight $g = g_f = \sum_{j=1}^N g_j e_j \in I_N^*$ by

$$g_N = -\frac{1}{N} \sum_{j=1}^{N-1} f_j, \quad g_j = f_j + g_N, \quad j = 1, \dots, N-1. \quad (17)$$

Then, by the assumption that $|f| \in N\mathbb{Z}$, g_j is an integer for every $j = 1, \dots, N$. It is easy to see that $\sum_{j=1}^N g_j = g(H_0) = 0$ and $g|_{\mathfrak{t}_N} = f|_{\mathfrak{t}_N} = \mu$. Thus, by (15), we have $\mu \in \Lambda^*$. Conversely, assume that $\mu \in \Lambda^*$. Then, by (15), there exists a $g = \sum_{j=1}^N g_j e_j \in I_N^*$ such that $g(H_0) = 0$ and $g|_{\mathfrak{t}_N} = \mu$. We define $f = f_g = \sum_{j=1}^{N-1} f_j e_j$ by $f_j = g_j - g_N$ for $j = 1, \dots, N-1$. Then, clearly, $f|_{\mathfrak{t}_N} = g|_{\mathfrak{t}_N} = \mu$, and we have $|f| = f(H_0) = -Ng_N \in N\mathbb{Z}$, which completes the proof. \square

The following Lemma 3.2 can be shown easily by using Lemma 3.1.

LEMMA 3.2. *Let $\mu \in L_N^*$, and let $f = f_\mu = \sum_{j=1}^{N-1} f_j e_j$ be the corresponding weight in I_N^* . Then, $\mu \in N\Lambda^*$ if and only if $f_j \in N\mathbb{Z}$ and $|f| \in N^2\mathbb{Z}$.*

LEMMA 3.3. *Let $\lambda \in P_N$ be a dominant weight. We denote, as before, by $|\lambda|$ the sum $\sum_{j=1}^{N-1} \lambda_j$ for the representative $f = f_\lambda = \sum_{j=1}^{N-1} \lambda_j e_j$ of λ . Assume that $|\lambda| = N\ell$ with a non-negative integer ℓ (so that, by Lemma 3.1, $\lambda \in \Lambda^*$). Then, the dominant weight λ satisfies the condition (K) if and only if there exists a permutation $w \in \mathfrak{S}_N$ such that*

$$\lambda_{w(j)} + j - w(j) - \ell \in N\mathbb{Z}, \quad j = 1, \dots, N,$$

where we set, as before, $\lambda_N = 0$. In the above condition, we can take the same permutation w as that in the condition (K).

Proof. First, assume that λ satisfies the condition (K), and let $w \in \mathfrak{S}_N$ denote the permutation in the condition (K). We set $\mu = w(\lambda + \rho_N) - \rho_N \in N\Lambda^*$. Then, one has

$$\mu = \sum_{j=1}^N [\lambda_{w(j)} + j - w(j)] e_j.$$

(Strictly speaking, the above expresses one of the representative of $\mu \in N\Lambda^*$.) We express the above weight in I_N^* as an element in $\text{span}_{\mathbb{Z}}(e_1, \dots, e_{N-1})$. We have $\mu = \sum_{j=1}^{N-1} \mu_j e_j$ with

$$\mu_j = \lambda_{w(j)} + j - w(j) - (\lambda_{w(N)} + N - w(N)). \quad (18)$$

Since $\mu \in N\Lambda^*$, by Lemma 3.2, we have $\mu_j \in N\mathbb{Z}$ and $|\mu| \in N^2\mathbb{Z}$. By (18), we have

$$|\mu| = |\lambda| - N\lambda_{w(N)} + Nw(N) - N^2 = -N(\lambda_{w(N)} + N - w(N) - \ell), \quad (19)$$

which shows that $\lambda_{w(N)} + N - w(N) - \ell \in N\mathbb{Z}$. Again by (18), we have

$$\mu_j \equiv \lambda_{w(j)} + j - w(j) \equiv 0 \pmod{N}.$$

Conversely, assume that there exists a $w \in \mathfrak{S}_N$ satisfying the condition in the lemma. Then, one can write

$$\lambda_{w(j)} + j - w(j) - \ell = Nc_j, \quad c_j \in \mathbb{Z}, \quad j = 1, \dots, N.$$

We set $\mu = w(\lambda + \rho_N) - \rho_N$. Then, (18) and (19) still hold for this μ , and which show that $\mu_j \in N\mathbb{Z}$ and $\sum \mu_j \in N^2\mathbb{Z}$. \square

3.1. Proof of Proposition 2.3. First of all, we assume that the dominant weight λ satisfies the condition (K). By Lemma 3.1 and the fact that $N\Lambda^* \subset \Lambda^*$, we have $|\lambda| \in N\mathbb{Z}$. We set $\ell = |\lambda|/N \in \mathbb{Z}$. Then, by Lemma 3.3, the permutation $w \in \mathfrak{S}_N$ in the condition (K) satisfies $\lambda_j + w^{-1}(j) - j - \ell \in N\mathbb{Z}$ for any $j = 1, \dots, N$, where we have replaced j by $w^{-1}(j)$ in the statement of Lemma 3.3. We write

$$\lambda_j + N - j - \ell = Na_j + N - w^{-1}(j).$$

Since $0 \leq N - w^{-1}(j) \leq N - 1$ are all distinct, the above equation shows that $N - w^{-1}(j)$ is the residue of $\lambda_j + N - j - \ell$ modulo N , and the residues are distinct. Thus the residues of $\lambda_j + N - j$'s are also distinct. Conversely, assume that the residues of $\lambda_j + N - j$'s modulo N are distinct. Denote their residues modulo N by c_j , $0 \leq c_j \leq N - 1$, $j = 1, \dots, N$. Then, one has

$$|\lambda| + \frac{N(N-1)}{2} \equiv \sum_{j=1}^N c_j \equiv \frac{N(N-1)}{2} \pmod{N},$$

which shows that $\ell := |\lambda|/N$ is a non-negative integer. We denote by $r(j)$, $0 \leq r(j) \leq N-1$ the residue of $\lambda_j + N - j - \ell$ modulo N . We define the permutation $w \in \mathfrak{S}_N$ by

$$w^{-1}(j) = N - r(j), \quad j = 1, \dots, N.$$

Now, it is easy to see that $\lambda_j + w^{-1}(j) - j - \ell \in N\mathbb{Z}$, and hence, by Lemma 3.3, λ satisfies the condition (K). By Theorem 2.2, we have

$$\chi_\lambda(a_N) = \text{sgn}(w^{-1}) = \text{sgn}(\sigma),$$

where $\sigma = w^{-1}$ is defined in Proposition 2.3. This completes the proof. \square

3.2. Proof of Theorem 1.2. This is a direct consequence of Proposition 2.3. In fact, let $\lambda(N)$ be a sequence of dominant weights such that $\lambda(N) + \rho_N$ tends weakly to a measure ρ_Y on the real line in the sense of (3). Note that we have $\lambda_1(N) \geq \lambda_2(N)$, and hence the weight $\mu(N) = \lambda(N) + e_1$ is also a dominant weight. Then we need to show that

- (i) the sequence of shifted dominant weights $\mu(N) + \rho_N = \lambda(N) + e_1 + \rho_N$ converges weakly to the same density ρ_Y as for the sequence $\lambda(N) + \rho_N$, and that
- (ii) $\chi_{\mu(N)}(a_N) = 0$.

To prove (ii), we observe that the residues of two components modulo N of the dominant weight $\mu(N) + \rho_N$ must coincide because the residues of the components of $\lambda(N) + \rho_N$ are all distinct, and the residues of the components of $\mu(N) + \rho_N$ differ from that of $\lambda(N) + \rho_N$ only in the first component. Thus, by Proposition 2.3, we have $\chi_{\mu(N)}(a_N) = 0$.

To prove (i), we let f be a compactly supported continuous function on the real line, and we denote by $d\rho_{\mu_N}$, resp. $d\rho_{\lambda_N}$ the measures in (2) for the corresponding irreducibles. Then we clearly have

$$\left| \int_{\mathbb{R}} f(x) [d\rho_{\mu_N} - d\rho_{\lambda_N}] \right| = \frac{1}{N} |f(\lambda_1(N)/N + 1) - f(\lambda_1(N)/N + 1 - 1/N)| \rightarrow 0, \quad N \rightarrow \infty.$$

Hence the sequence of the dominant weights $\mu(N) + \rho_N$ tends to the same limit as the limit of the sequence $\lambda(N) + \rho_N$. \square

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