A Class of Singular Continuous Functions

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A nondecreasing continuous function $F(x)$ on the real line is singular if its derivative $F'(x)$ is zero almost everywhere (a.e.). The most famous singular continuous function is Cantor's function discussed in every textbook on real analysis. In this short note we construct a class of singular continuous functions which includes Cantor's function as a
special case. Unlike Cantor’s function, most functions in this class are strictly increasing on the interval [0, 1].

We construct these singular functions as the distribution functions of certain random variables. We also give an explicit, but less instructive, formula for these functions. Readers not familiar with probability theory may be more comfortable with the explicit formula.

Let \( N \geq 2 \) be a positive integer and \( \mathbf{p} = [p_0, \ldots, p_{N-1}] \) be an \( N \)-dimensional probability vector, namely,

\[
p_i \geq 0, \quad i = 0, \ldots, N - 1; \quad \sum_{i=0}^{N-1} p_i = 1.
\]

We exclude the trivial case where one of the \( p_i \)'s is equal to 1. Next let \( X_n, n = 1, 2, \ldots \) be a sequence of independent, identically distributed random variables with the common distribution

\[
P \left[ X_1 = i \right] = p_i, \quad i = 0, \ldots, N - 1.
\]

Define a new random variable \( X \) with values in the interval [0, 1]:

\[
X = \sum_{n=1}^{\infty} \frac{X_n}{N^n}.
\]

Thus \( X \) is a number in [0, 1] whose digits in base \( N \) expansion \( X = 0.X_1X_2 \cdots \) are chosen randomly and independently according to the probability vector \( \mathbf{p} \). Let \( F(x; \mathbf{p}) \) be the distribution function of \( X \),

\[
F(x; \mathbf{p}) = P \left[ X \leq x \right].
\]

We now derive an explicit formula for \( F(x; \mathbf{p}) \). In the following we simply write \( F(x) \) instead of \( F(x; \mathbf{p}) \) if no confusion is possible.

It is clear that \( F(x) = 0 \) for \( x \leq 0 \) and \( F(x) = 1 \) for \( x \geq 1 \). For \( 0 < x < 1 \), suppose that \( x = 0.x_1x_2 \cdots \) is the expansion of \( x \) in base \( N \). For definitiveness, when two expansions are possible, for example 0.130 = 0.12N − 1 (\( k \) means repeating the digit \( k \) indefinitely), unless indicated otherwise we always take the nonterminating one, namely, the one with only the digit \( N - 1 \) from some digit on. Let \( x^n = 0.x_1x_2 \cdots x_n \) be the truncation of \( x \) at the \( n \)th place. Then since \( P \left[ X = x \right] = P \left[ X \leq 0 \right] = 0 \), we have

\[
F(x) = \sum_{n=0}^{\infty} P \left[ x^n < X \leq x^{n+1} \right].
\]

Now \( x^n < X \leq x^{n+1} \) means that \( X \) must have the form \( X = 0.x_1x_2 \cdots x_nX_{n+1}X_{n+2} \cdots \), where \( X_{n+1} \) is equal to one of the digits 0, 1, \ldots, \( N - 1 \). Note that we may assume that the expansion \( X = 0.X_1X_2 \cdots \) is always nonterminating, for the probability that \( X \) has a terminating expansion is zero. Thus

\[
P \left[ x^n < X \leq x^{n+1} \right] = p_{x_1} \cdot \cdots \cdot p_{x_n} \left( p_0 + \cdots + p_{N-1} \right).
\]
In the above inequality, the sum between the parentheses should be set equal to zero if \( x_{n+1} = 0 \). We now obtain

\[
F(x; p) = \sum_{n=0}^{\infty} p_{x_n} \cdots p_{x_1} \left( p_0 + \cdots + p_{x_n-1} \right).
\]

This is an explicit formula for the function \( F(x; p) \). Incidentally, the above explicit representation works when \( x \) is expressed in its terminating expansion as well. The special case \( N = 3, p = \left[ \frac{1}{3}, 0, \frac{1}{3} \right] \) gives Cantor's function.

We will prove two properties of these functions.

(I) \( F(x; p) \) is strictly increasing unless one of the \( p_i \)'s is equal to zero, and is Hölder continuous with the exponent

\[
\alpha = \frac{\log \frac{1}{r}}{\log N},
\]

where \( r = \max \{ p_0, \ldots, p_{N-1} \} \).

(II) \( F(x; p) \) is singular continuous except when \( p \) is the uniform distribution:

\[
p = \left[ \frac{1}{N}, \ldots, \frac{1}{N} \right],
\]

in which case \( F(x; p) = x \) on \([0, 1]\).

**Proof of (I).** If \( x, y \in [0, 1] \) have the same first \( n \) digits, then from the explicit formula for \( F(x) \) we have

\[
|F(x) - F(y)| \leq \sum_{i=n}^{\infty} r^i \leq \frac{r^n}{1-r}.
\]

Let \( x, y \) be two distinct numbers in \([0, 1]\). Pick a nonnegative \( n \) such that \( N^{-(n+2)} \leq |x - y| < N^{-(n+1)} \). Then in the nonterminating expansions of \( x \) and \( y \) either they have the same first \( n \) digits or \( x^n = y^n + N^{-n} \). In the latter case \( y^n = 0.x_1x_2\cdots x_nN^{-1} \) and \( x^n \) have the same first \( n \) digits. Therefore we have

\[
|F(x) - F(y)| \leq |F(x) - F\left(x^n\right)| + |F\left(x^n\right) - F\left(y^n\right)| + |F\left(y^n\right) - F(y)|
\]

\[
\leq \frac{3r^n}{1-r}.
\]

It follows that for any \( x, y \) in \([0,1]\) with \(|x - y| \leq N^{-1}\) we have

\[
|F(x) - F(y)| \leq \frac{3}{r^2(1-r)}|x - y|^9.
\]

This shows that \( F(x) \) is Hölder continuous with exponent \( \alpha \).
Let \( x = 0.x_1x_2\ldots \) (in base \( N \)) be a point in \([0, 1]\). Let \( x^n = 0.x_1x_2\ldots x_n \) as before. Then
\[
x^n < x \leq x^n + \frac{1}{N^n}.
\]
Writing \( x^n + N^{-n} \) as \( 0.x_1\ldots x_nN^{-1} \), from the explicit formula for \( F(x) \) we have
\[
F(x^n + N^{-n}) - F(x^n) = p_{x_1} \cdots p_{x_n}.
\]
This shows that \( F \) is strictly increasing on \((0, 1)\) if none of the \( p_i \)'s is equal to zero.

**Proof of (II).** From the above equality we see that
\[
D_n F(x) \overset{\text{def}}{=} \prod_{i=1}^{n} \left[ F(x^n + N^{-n}) - F(x^n) \right] = \prod_{i=1}^{n} p_{x_i} = \prod_{i=1}^{n} p_{x_i}.
\]
Since \( F(x) \), being a monotone function, is a.e. differentiable, we have a.e.,
\[
\lim_{n \to \infty} D_n F(x) = F'(x) \text{ and } F'(x) \text{ is finite.}
\]
It follows that
\[
F'(x) = \prod_{n=1}^{\infty} N p_{x_n}, \quad \text{a.e.}
\]
If all of \( p_i \)'s are equal to \( N^{-1} \), then \( F'(x) = 1 \) a.e. on \([0, 1]\). We have \( F(x) = x \) on \([0, 1]\).

If at least one of the \( p_i \)'s is not equal to \( N^{-1} \), say, \( p_1 \neq N^{-1} \), then for those \( x \) whose expansion has infinitely many 1’s the general term \( N p_{x_n} \) of the above infinite product does not converge to 1. Therefore, by an elementary theorem on infinite products, the value of the infinite product, if it exists and is finite, must be 0. On the other hand, the reader can verify easily that the numbers whose expansions have only finitely many 1’s form a set of Lebesgue measure zero. Therefore we have proved \( F'(x) = 0 \) a.e. and \( F(x) \) is a singular continuous function.

For readers familiar with the strong law of large numbers in probability theory the following argument may be more appealing. By the strong law of large numbers, almost all numbers in \([0, 1]\) are normal in base \( N \), namely, if \( s_n(x; i) \) is the number of the digit \( i \) among the first \( n \) digits of the expansion of \( x \), then for almost all \( x \in [0, 1] \),
\[
\frac{s_n(x; i)}{n} \to \frac{1}{N} \quad \text{as } n \to \infty.
\]
It follows that, if at least one of the \( p_i \)'s is not equal to \( N^{-1} \), then for almost all \( x \in [0, 1] \)
\[
\sqrt[n]{D_n F(x)} = \prod_{i=0}^{N-1} (N p_i) s_n(x; i) \to N \sqrt[n]{p_0 \cdots p_{N-1}} < 1.
\]
This implies immediately \( F'(x) = 0 \), a.e.

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