1. Determine whether the following series converges or diverges. If it converges, find its sum.

\[ \sum_{n=1}^{\infty} \sin^n 1 \]

Answer:

It is a geometric series of ratio \( r = \sin 1 \). Since \(|r| < 1\), the series converges. The sum is

\[ \sum_{n=1}^{\infty} \sin^n 1 = \frac{\sin 1}{1 - \sin 1} \]
2. Find the Taylor series of \( \ln x \) at \( a = 1 \).

\[
\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-1)^n}{n} = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \ldots
\]
3. Determine if the following series converges or diverges:

\[ \sum_{n=1}^{\infty} \frac{e^{1/n}}{n} \]

Answer:

We have that:

\[ \sum_{n=1}^{\infty} \frac{e^{1/n}}{n} > \sum_{n=1}^{\infty} \frac{1}{n}, \]

so by the comparison test, the series *diverges* (recall that the harmonic series diverges).
4. Determine if the following series converges absolutely, converges conditionally, or diverges:

\[ \sum_{n=0}^{\infty} \frac{(-10)^n}{n!} \]

**Answer:**

We use the ratio test:

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{10^{n+1}/(n+1)!}{10^n/n!} = \lim_{n \to \infty} \frac{10}{n + 1} = 0 < 1,
\]

hence the series converges absolutely.
5. Find a power series for the following function and find its radius of convergence: \( f(x) = \frac{x}{1-x} \)

Answer:

Since \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \), we have:

\[
\frac{x}{1-x} = x \sum_{n=0}^{\infty} x^n = x (1 + x + x^2 + x^3 + \cdots) = x + x^2 + x^3 + x^4 + \cdots = \sum_{n=1}^{\infty} x^n
\]

In order to find its radius of convergence, we use the ratio test:

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n \to \infty} |x| = |x|
\]

There is convergence for \( \rho < 1 \), i.e., \( |x| < 1 \). At the endpoints we get the series

\[
\sum_{n=1}^{\infty} (-1)^n \quad \text{and} \quad \sum_{n=1}^{\infty} 1^n
\]

respectively, whose \( n \)-th terms do not tend to zero, so they do not converge. Hence, the interval of convergence is:

\( I = (-1, 1) \),

not including the endpoints.
6. Use power series to evaluate the following limit:

\[
\lim_{x \to 1} \frac{\ln(x^2)}{x - 1} = \]

**Answer:**

\[
\lim_{x \to 1} \frac{\ln(x^2)}{x - 1} = \lim_{x \to 1} \frac{(x^2 - 1) - (x^2 - 1)^2/2 + (x^2 - 1)^3/3 - \cdots}{x - 1} = \lim_{x \to 1} \{(x + 1) - (x + 1)(x^2 - 1)/2 + (x + 1)(x^2 - 1)^2/3 - \cdots\} = 2
\]

An alternative (and simpler) way consists of using \(\ln(x^2) = 2 \ln x\):

\[
\lim_{x \to 1} \frac{\ln(x^2)}{x - 1} = 2 \lim_{x \to 1} \frac{\ln x}{x - 1} = 2 \lim_{x \to 1} \frac{(x - 1) - (x - 1)^2/2 + (x - 1)^3/3 - \cdots}{x - 1} = 2 \lim_{x \to 1} \{(1 - (x - 1)/2 + (x - 1)^2/3 - \cdots\} = 2
\]