1. (Logic)

1. Prove the following logical equivalence by using Laws of Logic (Algebra of Propositions):

\[(p \land q) \rightarrow r \iff (p \rightarrow r) \lor (q \rightarrow r)\].

(Assume that \(\rightarrow\) is defined by \(p \rightarrow q \iff \neg p \lor q\).)

2. For each of the following quantified statements find a model and a countermodel (if any exists):

(a) \(\exists x \exists y \forall z [(z = x) \lor (z = y)]\).
(b) \(\exists x \exists y \exists z [(x \neq y) \land (x \neq z) \land (y \neq z)]\).
(c) \(\forall x \exists y (x = y)\).
(d) \(\exists x \forall y (x = y)\).

**Solution:**

1. \((p \land q) \rightarrow r \quad \text{(Def. of } \rightarrow)\)
   \[\iff \neg (p \land q) \lor r \quad \text{(DeMorgan’s)}\]
   \[\iff \neg p \lor \neg q \lor r \quad \text{(Idempotent)}\]
   \[\iff \neg p \lor \neg q \lor r \lor r \quad \text{(Associative)}\]
   \[\iff (p \rightarrow r) \lor (q \rightarrow r) \quad \text{(Def. of } \rightarrow)\]

2. (a) Model: \(\{0, 1\}\) (or any set with at most two elements).
   Countermodel: \(\mathbb{Z}\) (or any set with more than two elements).
(b) Model: \(\{0, 1, 2\}\) (or any set with at least three elements).
   Countermodel: \(\{0\}\) (or any set with less than three elements).
(c) Model: Any (the statement is always true).
   Countermodel: There is none.
(d) Model: \(\{0\}\) (or any set with exactly one element).
   Countermodel: \(\mathbb{Z}\) (or any set with at least two elements).
2. (Sets) Let $A, B, C,$ be the following sets: $A = \{a, b, c\}$, $B = \{x \in \mathbb{Z} \mid 0 \leq x < 3\}$, $C = \{x \in \mathbb{Z} \mid 0 < x \leq 3\}$. Find the following sets (list their elements):

1. $B \cap C =$
2. $A \times (B \cap C) =$
3. $A \times B =$
4. $A \times C =$
5. $(A \times B) \cap (A \times C) =$

Solution:

1. $B \cap C = \{1, 2\}$
2. $A \times (B \cap C) = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$
3. $A \times B = \{(a, 0), (b, 0), (c, 0), (a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$
4. $A \times C = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2), (a, 3), (b, 3), (c, 3)\}$
5. $(A \times B) \cap (A \times C) = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$
3. (Operations) On $\mathbb{R}^2$ we define the following operation:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

1. Prove that $(\mathbb{R}^2, +)$ is a commutative group.

2. Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined $f(x, y) = x$ is a group-homomorphism from $(\mathbb{R}^2, +)$ to $(\mathbb{R}, +)$.

Solution:

1. (a) Commutative property:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = (x_2, y_2) + (x_1, y_1).$$

(b) Associative property:

$$[(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = (x_1 + x_2, y_1 + y_2) + (x_3, y_3) = (x_1 + x_2 + x_3, y_1 + y_2 + y_3),$$

$$(x_1, y_1) + [(x_2, y_2) + (x_3, y_3)] = (x_1, y_1) + (x_2 + x_3, y_2 + y_3) = (x_1 + x_2 + x_3, y_1 + y_2 + y_3).$$

Hence:

$$[(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)].$$

(c) Identity element. The identity element is $(0, 0)$:

$$(x, y) + (0, 0) = (x + 0, y + 0) = (x, y).$$

(d) Inverse element: The inverse element of $(x, y)$ is $(-x, -y)$:

$$(x, y) + (-x, -y) = (x - x, y - y) = (0, 0).$$

2. $f((x_1, y_1) + (x_2, y_2)) = f(x_1 + x_2, y_1 + y_2) = x_1 + x_2.$

$\quad f(x_1, y_1) + f(x_2, y_2) = x_1 + x_2.$

Hence: $f((x_1, y_1) + (x_2, y_2)) = f(x_1, y_1) + f(x_2, y_2).$
4. (Relations) On $\mathbb{R}^2$ we define the relation

$$(x_1, y_1) \sim (x_2, y_2) \iff 2x_1 + 3y_1 = 2x_2 + 3y_2.$$ 

Prove that $\sim$ is an equivalence relation.

**Solution:**

1. Reflexive: $2x + 3y = 2x + 3y \Rightarrow (x, y) \sim (x, y)$.

2. Symmetric: $(x_1, y_1) \sim (x_2, y_2) \Rightarrow 2x_1 + 3y_1 = 2x_2 + 3y_2 \Rightarrow 2x_2 + 3y_2 = 2x_1 + 3y_1 \Rightarrow (x_2, y_2) \sim (x_1, y_1)$.

3. Transitive:

$$(x_1, y_1) \sim (x_2, y_2) \Rightarrow 2x_1 + 3y_1 = 2x_2 + 3y_2 \quad \text{and} \quad (x_2, y_2) \sim (x_3, y_3) \Rightarrow 2x_2 + 3y_2 = 2x_3 + 3y_3 \Rightarrow 2x_1 + 3y_1 = 2x_3 + 3y_3 \Rightarrow (x_1, y_1) \sim (x_3, y_3).$$
5. (Functions) Let $A$ be the set $A = \{0, 1, 2, 3, 4\}$. Let $f, g : A \rightarrow A$ be defined in the following way: $f(0) = 1$, $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 0$; $g(0) = 1$, $g(1) = 0$, $g(2) = 2$, $g(3) = 3$, $g(4) = 4$. A convenient way of representing those functions consists of listing between parenthesis the images of 0, 1, 2, 3, 4 in this order, so that $f = (1, 2, 3, 4, 0)$, $g = (1, 0, 2, 3, 4)$.

1. Find $g \circ f$, $f \circ g$, $f^{-1}$, $g^{-1}$, $(g \circ f)^{-1}$, $(f \circ g)^{-1}$.

2. Let $h : A \rightarrow A$ be the function $h = (0, 2, 1, 3, 4)$. Find $h \circ f$ and $f \circ h$. Compare to the previously found compositions and write $h$ as a suitable composition of $f$ and $g$.

Solution:

1. $g \circ f = (0, 2, 3, 4, 1)$.
   $f \circ g = (2, 1, 3, 4, 0)$.
   $f^{-1} = (4, 0, 1, 2, 3)$.
   $g^{-1} = (1, 0, 2, 3, 4)$.
   $(g \circ f)^{-1} = (0, 4, 1, 2, 3)$.
   $(f \circ g)^{-1} = (4, 1, 0, 2, 3)$.

2. $h \circ f = (2, 1, 3, 4, 0)$.
   $f \circ h = (1, 3, 2, 4, 0)$.
   $h \circ f = f \circ g$, hence: $h = f \circ g \circ f^{-1}$.
6. (Counting)

(a) How many non negative integer solutions does the following equation have?

\[ x_1 + x_2 + x_3 + x_4 = 10. \]

(b) How many of those solutions are strictly positive?

(c) How many non negative solutions consists of even numbers only?

Solution:

(a) \( \binom{4+10-1}{10} = \binom{13}{10} = 286. \)

(b) Calling \( x_1 = y_1 + 1, x_2 = y_2 + 1, x_3 = y_3 + 1, x_4 = y_4 + 1, \) the equation becomes:

\[ y_1 + y_2 + y_3 + y_4 = 10 - 4 = 6. \]

Its non negative solutions correspond to strictly positive solutions to the original equation. Their number is \( \binom{4+6-1}{6} = \binom{9}{6} = 84. \)

(c) Calling \( x_1 = 2z_1, x_2 = 2z_2, x_3 = 2z_3, x_4 = 2z_4, \) the equation becomes:

\[ z_1 + z_2 + z_3 + z_4 = 5. \]

Its non negative solutions correspond to even non negative solutions to the original equation. Their number is \( \binom{4+5-1}{5} = \binom{8}{5} = 56. \)