2.3. Arc Length, Parametric Curves

2.3.1. Parametric Curves. A parametric curve can be thought of as the trajectory of a point that moves through the plane with coordinates \((x, y) = (f(t), g(t))\), where \(f(t)\) and \(g(t)\) are functions of the parameter \(t\). For each value of \(t\) we get a point of the curve. Example: A parametric equation for a circle of radius 1 and center \((0,0)\) is:

\[
x = \cos t, \quad y = \sin t.
\]

The equations \(x = f(t), \ y = g(t)\) are called parametric equations.

Given a parametric curve, sometimes we can eliminate \(t\) and obtain an equivalent non-parametric equation for the same curve. For instance \(t\) can be eliminated from \(x = \cos t, \ y = \sin t\) by using the trigonometric relation \(\cos^2 t + \sin^2 t = 1\), which yields the (non-parametric) equation for a circle of radius 1 and center \((0,0)\):

\[
x^2 + y^2 = 1.
\]

Example: Find a non-parametric equation for the following parametric curve:

\[
x = t^2 - 2t, \quad y = t + 1.
\]

Answer: We eliminate \(t\) by isolating it from the second equation:

\[
t = (y - 1),
\]

and plugging it in the first equation:

\[
x = (y - 1)^2 - 2(y - 1).
\]

i.e.:

\[
x = y^2 - 4y + 3,
\]

which is a parabola with horizontal axis.

2.3.2. Arc Length. Here we describe how to find the length of a smooth arc. A smooth arc is the graph of a continuous function whose derivative is also continuous (so it does not have corner points).

If the arc is just a straight line between two points of coordinates \((x_1, y_1), (x_2, y_2)\), its length can be found by the Pythagorean theorem:

\[
L = \sqrt{(\Delta x)^2 + (\Delta y)^2},
\]

where \(\Delta x = x_2 - x_1\) and \(\Delta y = y_2 - y_1\).
More generally, we approximate the length of the arc by inscribing a polygonal arc (made up of straight line segments) and adding up the lengths of the segments. Assume that the arc is given by the parametric functions \( x = f(t), \ y = g(t), \ a \leq t \leq b. \)

We divide the interval into \( n \) subintervals of equal length. The corresponding points in the arc have coordinates \((f(t_i), g(t_i))\), so two consecutive points are separated by a distance equal to

\[
L_i = \sqrt{(f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2}.
\]

We have \( \Delta t = t_i - t_{i-1} = (b - a)/n. \) On the other hand, by the mean value theorem

\[
f(t_i) - f(t_{i-1}) = f'(t_i^*) \Delta t \\
g(t_i) - f(t_{i-1}) = g'(t_i^*) \Delta t
\]

for some \( t_i^* \) in \( [t_{i-1}, t_i] \). Hence

\[
L_i = \sqrt{(f'(x_i^*) \Delta t)^2 + (g'(t_i^*) \Delta t)^2} = \sqrt{(f'(t_i^*))^2 + (g'(t_i^*))^2} \Delta t.
\]

The total length of the arc is

\[
L \approx \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \sqrt{(f'(t_i^*))^2 + (g'(t_i^*))^2} \Delta t,
\]

which converges to the following integral as \( n \to \infty \):

\[
L = \int_{a}^{b} \sqrt{(f'(t))^2 + (g'(t))^2} \, dt.
\]

This formula can also be expressed in the following (easier to remember) way:

\[
L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

The last formula can be obtained by integrating the length of an “infinitesimal” piece of arc

\[
ds = \sqrt{(dx)^2 + (dy)^2} = dt \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.
\]

Example: Find the arc length of the curve \( x = t^2, \ y = t^3 \) between \((1, 1)\) and \((4, 8)\).
Answer: The given points correspond to the values \( t = 1 \) and \( t = 2 \) of the parameter, so:

\[
L = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[
= \int_1^2 \sqrt{4t^2 + 9t^4} \, dt
\]

\[
= \int_1^2 t\sqrt{4 + 9t^2} \, dt
\]

\[
= \frac{1}{18} \int_{13}^{40} \sqrt{u} \, du \quad (u = 4 + 9t^2)
\]

\[
= \frac{1}{27} \left[ 40^{3/2} - 13^{3/2} \right]
\]

\[
= \frac{1}{27} (80\sqrt{10} - 13\sqrt{13})
\]

In cases when the arc is given by an equation of the form \( y = f(x) \) or \( x = f(x) \) the formula becomes:

\[
L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

or

\[
L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \, dy
\]

Example: Find the length of the arc defined by the curve \( y = x^{3/2} \) between the points \((0, 0)\) and \((1, 1)\).
Answer:

\[ L = \int_0^1 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_0^1 \sqrt{1 + \left( \left( x^{3/2} \right)^2 \right) dx} \]

\[ = \int_0^1 \sqrt{1 + \left( \frac{3x^{1/2}}{2} \right)^2} \, dx = \int_0^1 \sqrt{1 + \frac{9x}{4}} \, dx \]

\[ = \left[ \frac{1}{27} (4 + 9x^{3/2}) \right]^1_0 = \frac{1}{27} (13^{3/2} - 8) \].