2.4. Average Value of a Function (Mean Value Theorem)

2.4.1. Average Value of a Function. The average value of finitely many numbers \( y_1, y_2, \ldots, y_n \) is defined as

\[
y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}.
\]

The average value has the property that if each of the numbers \( y_1, y_2, \ldots, y_n \) is replaced by \( y_{\text{ave}} \), their sum remains the same:

\[
y_1 + y_2 + \cdots + y_n = (n \text{ times}) \underbrace{y_{\text{ave}} + y_{\text{ave}} + \cdots + y_{\text{ave}}}.
\]

Analogously, the average value of a function \( y = f(x) \) in the interval \([a, b]\) can be defined as the value of a constant \( f_{\text{ave}} \) whose integral over \([a, b]\) equals the integral of \( f(x) \):

\[
\int_a^b f(x) \, dx = \int_a^b f_{\text{ave}} \, dx = (b - a) f_{\text{ave}}.
\]

Hence:

\[
f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x) \, dx.
\]

2.4.2. The Mean Value Theorem for Integrals. If \( f \) is continuous on \([a, b]\), then there exists a number \( c \) in \([a, b]\) such that

\[
f(c) = f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x) \, dx,
\]

i.e.,

\[
\int_a^b f(x) \, dx = f(c)(b - a).
\]

Example: Assume that in a certain city the temperature (in °F) \( t \) hours after 9 A.M. is represented by the function

\[
T(t) = 50 + 14 \sin \frac{\pi t}{12}.
\]

Find the average temperature in that city during the period from 9 A.M. to 9 P.M.
Answer:

\[
T_{\text{ave}} = \frac{1}{12} - 0 \int_{0}^{12} \left( 50 + 14 \sin \frac{\pi t}{12} \right) dt
\]

\[
= \frac{1}{12} \left[ 50t - \frac{14 \cdot 12}{\pi} \cos \frac{\pi t}{12} \right]_{0}^{12}
\]

\[
= \frac{1}{12} \left\{ \left( 50 \cdot 12 - \frac{168}{\pi} \cos \frac{12\pi}{12} \right) - \left( 50 \cdot 0 - \frac{168}{\pi} \cos 0 \right) \right\}
\]

\[
= 50 + \frac{28}{\pi} \approx 58.9 .
\]