1.3. The Fundamental Theorem of Calculus

1.3.1. The Fundamental Theorem of Calculus. The Fundamental Theorem of Calculus (FTC) connects the two branches of calculus: differential calculus and integral calculus. It says the following:

Suppose $f$ is continuous on $[a, b]$. Then:

(1) The function

$$ g(x) = \int_a^x f(t) \, dt $$

is an antiderivative of $f$, i.e., $g'(x) = f(x)$.

(2) (Evaluation Theorem) If $F$ is an antiderivative of $f$, i.e. $F'(x) = f(x)$, then

$$ \int_a^b f(x) \, dx = F(b) - F(a). $$

The two parts of the theorem can be rewritten like this:

(1) $\frac{d}{dx} \int_a^x f(t) \, dt = f(x)$.

(2) $\int_a^b F'(x) \, dx = F(b) - F(a)$.

So the theorem states that integration and differentiation are inverse operations, i.e., the derivative of an integral of a function yields the original function, and the integral of a derivative also yields the function originally differentiated (up to a constant).

Example: Find $\frac{d}{dx} \int_0^x t^3 \, dt$.

Answer: We solve this problem in two ways. First directly:

$$ g(x) = \int_0^x t^3 \, dt = \left[ \frac{t^4}{4} \right]_0^x = \frac{x^4}{4} - \frac{0^4}{4} = \frac{x^4}{4}, $$

hence

$$ g'(x) = \frac{8x^3}{4} = 2x^3. $$
Second, using the FTC:

\[ h(u) = \int_0^u t^3 \, dt \Rightarrow h'(u) = u^3. \]

Now we have \( g(x) = h(x^2) \), hence (using the chain rule):

\[ g'(x) = h'(x^2) \cdot 2x = (x^2)^3 \cdot 2x = 2x^7. \]