PUTNAM TRAINING
CALCULUS

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REMARK. This is a list of exercises on calculus. —Miguel A. Lerma

EXERCISES

1. Believe it or not the following function is constant in an interval \([a, b]\). Find that interval and the constant value of the function.
   \[ f(x) = \sqrt{x + 2\sqrt{x - 1} + \sqrt{x - 2\sqrt{x - 1}}}. \]

2. Find the value of the following infinitely nested radical
   \[ \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}. \]

3. (Putnam 1995) Evaluate
   \[ \sqrt{\frac{2207 - \frac{1}{1}}{2207 - \frac{1}{2207 - \cdots}}}. \]
   Express your answer in the form \((a + b\sqrt{c})/d\), where \(a, b, c, d\), are integers.

4. (Putnam 1992) Let \( f \) be an infinitely differentiable real-valued function defined on the real numbers. If
   \[ f\left(\frac{1}{n}\right) = \frac{n^2}{n^2 + 1}, \quad n = 1, 2, 3, \ldots \]
   compute the values of the derivatives \( f^{(k)}(0) \), \( k = 1, 2, 3, \ldots \).

5. Compute \( \lim_{n \to \infty} \left\{ \frac{1}{n} + \frac{1}{n + 1} + \frac{1}{n + 2} + \cdots + \frac{1}{2n - 1} \right\} \).

6. Compute \( \lim_{n \to \infty} \left\{ \prod_{k=1}^{n} \left( 1 + \frac{k}{n} \right) \right\}^{1/n} \).

7. (Putnam 1997) Evaluate
   \[ \int_{0}^{\infty} \left( x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left( 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) \, dx \].
8. (Putnam 1990) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt{n} - \sqrt{m}$ ($n, m = 0, 1, 2, \ldots$)? (In other words, is it possible to find integers $n$ and $m$ such that $\sqrt{n} - \sqrt{m}$ is as close as we wish to $\sqrt{2}$?)

9. (Leningrad Mathematical Olympiad, 1988) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with $f(x) \cdot f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.

10. Let $f : [0, 1] \to \mathbb{R}$ continuous, and suppose that $f(0) = f(1)$. Show that there is a value $x \in [0, 1998/1999]$ satisfying $f(x) = f(x + 1/1999)$.

11. For which real numbers $c$ is $(e^x + e^{-x})/2 \leq e^{cx^2}$ for all real $x$?

12. Does there exist a positive sequence $a_n$ such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 1/(n^2 a_n)$ are convergent?
Hints

1. $\sqrt{u^2} = |u|$.

2. Find the limit of the sequence $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2 + a_n}$ ($n \geq 1$).

3. Call the limit $L$. Find some equation verified by $L$.

4. Justify that the desired derivatives must coincide with those of the function $g(x) = 1/(1 + x^2)$.

5. Compare the sum to some integral of the form $\int_a^b \frac{1}{x} \, dx$.

6. Take logarithms. Interpret the resulting expression as a Riemann sum.

7. Interpret the first series as a Maclaurin series. Interchange integration and summation with the second series (don’t forget to justify why the interchange is “legitimate”.)

8. In fact any real number $r$ is the limit of a sequence of numbers of the form $\sqrt[n]{n} - \sqrt[m]{m}$. We want $r \approx \sqrt[n]{n} - \sqrt[m]{m}$, i.e., $r + \sqrt[m]{m} \approx \sqrt[n]{n}$. Note that $\sqrt[n]{n} + 1 - \sqrt[n]{n} \to 0$ as $n \to \infty$.

9. If $y \in f(\mathbb{R})$ what is $f(y)$?

10. Consider the function $g(x) = f(x) - f(x + 1/999)$. Use the intermediate value theorem.


12. If they were convergent their sum would be convergent too.
1. The solution is based on the fact that $\sqrt{u^2} = |u|$. Letting $u = 1 \pm \sqrt{x-1}$ we have that $u^2 = x \pm 2\sqrt{x-1}$, hence the given function turns out to be:

\[
    f(x) = |1 + \sqrt{x-1}| + |1 - \sqrt{x-1}|
\]

Defined for $x \geq 1$.

The expression $1 + \sqrt{x-1}$ is always positive, hence $|1 + \sqrt{x-1}| = 1 + \sqrt{x-1}$. On the other hand $|1 - \sqrt{x-1}| = 1 - \sqrt{x-1}$ if $1 - \sqrt{x-1} \geq 0$ and $|1 - \sqrt{x-1}| = -1 + \sqrt{x-1}$ if $1 - \sqrt{x-1} < 0$, hence

\[
    f(x) = \begin{cases} 
        1 + \sqrt{x-1} + 1 - \sqrt{x-1} = 2 & \text{if } 1 - \sqrt{x-1} \geq 0 \\
        1 + \sqrt{x-1} - 1 + \sqrt{x-1} = 2\sqrt{x-1} & \text{if } 1 - \sqrt{x-1} < 0
    \end{cases}
\]

So the function is equal to 2 if $1 - \sqrt{x-1} \geq 0$, which happens for $1 \leq x \leq 2$. So $f(x) = 2$ (constant) in $[1, 2]$.

2. The desired value is the limit of the following sequence:

\[
    a_1 = \sqrt{2} \\
    a_2 = \sqrt{2 + \sqrt{2}} \\
    a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \\
    \ldots
\]

defined by the recursion $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ ($n \geq 1$).

First we must prove that the given sequence has a limit. To that end we prove

1. The sequence is bounded. More specifically, $0 < a_n < 2$ for every $n = 1, 2, \ldots$.

   This can be proved by induction. It is indeed true for $a_1 = \sqrt{2}$. Next, if we assume that $0 < a_n < 2$, then $0 < a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = \sqrt{4} = 2$.

2. The sequence is increasing. In fact: $a_{n+1}^2 = 2 + a_n > a_n + a_n = 2a_n > a_n^2$, hence $a_{n+1} > a_n$.

According to the *Monotonic Sequence Theorem*, every bounded monotonic (increasing or decreasing) sequence has a limit, hence $a_n$ must have in fact a limit $L = \lim_{n \to \infty} a_n$.

Now that we know that the sequence has a limit $L$, by taking limits in the recursive relation $a_{n+1} = \sqrt{2 + a_n}$, we get $L = \sqrt{2 + L}$, hence $L^2 - L - 2 = 0$, so $L = 2$ or $-1$.

Since $a_n > 0$ then $L \geq 0$, hence $L = 2$. Consequently:

\[
    \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}} = 2}.
\]

3. We will prove that the answer is $(3 + \sqrt{5})/2$.

The value of the infinite continued fraction is the limit $L$ of the sequence defined recursively $x_0 = 2207$, $x_{n+1} = 2207 - 1/x_n$, which exists because the sequence is decreasing (induction). Taking limits in both sides we get that $L = 2007 - 1/L$. Since
7. The series on the left is $xe^{-x^2/2}$. Since the terms of the second sum are non-negative, we can interchange the sum and integral:

$$
\int_0^\infty xe^{-x^2/2} \sum_{n=0}^\infty \frac{x^{2n}}{2^n(n!)^2} \, dx = \sum_{n=0}^\infty \int_0^\infty xe^{-x^2/2} \frac{x^{2n}}{2^n(n!)^2} \, dx
$$

$x_n > 1$ for all $n$ (also proved by induction), we have that $L \geq 1$. If we call the answer $r$ we have $r^8 = L$, so $r^8 + 1/r^8 = 2207$. Then $(r^4 + 1/r^4)^2 = r^8 + 2 + 1/r^8 = 2 + 2207 = 2209$, hence $r^4 + 1/r^4 = \sqrt{2209} = 47$. Analogously, $(r^2 + 1/r^2)^2 = r^4 + 2 + 1/r^4 = 2 + 47 = 49$, so $r^2 + 1/r^2 = \sqrt{49} = 7$. And $(r + 1/r)^2 = r^2 + 1 + 1/r^2 = 2 + 7 = 9$, so $r + 1/r = \sqrt{9} = 3$. From here we get $r^2 - 3r + 1 = 0$, hence $r = (3 + \sqrt{5})/2$, but $r = L^{1/8} \geq 1$, so $r = (3 + \sqrt{5})/2$.

4. That function coincides with $g(x) = 1/(1 + x^2)$ at the points $x = 1/n$, and the derivatives of $g$ at zero can be obtained from its Maclaurin series $g(x) = 1 - x^2 + x^4 - x^6 + \cdots$, namely $g^{(2k)}(0) = (-1)^k k!$ and $g^{(2k+1)}(0) = 0$. In order to prove that the result applies to $f$ too we have to study their difference $h(x) = f(x) - g(x)$.

We have that $h(x)$ is infinitely differentiable. Also $h(1/n) = 0$ for $n = 1, 2, 3, \ldots$, hence $h(0) = \lim_{n \to \infty} h(1/n) = 0$. By Rolle’s theorem, $h'(x)$ has zeros between the zeros of $h(x)$, hence $h'(0)$ is the limit of a sequence of zeros, so $h'(0) = 0$. The same is true about all derivatives of $h$ at zero. This implies that $f^{(k)}(0) = g^{(k)}(0)$ for every $k = 1, 2, 3, \ldots$, hence $f^{(2k)}(0) = (-1)^k k!$ and $f^{(2k+1)}(0) = 0$.

5. By looking at the graph of the function $y = 1/x$ we can see that

$$
\int_n^{2n} \frac{1}{x} \, dx < \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} < \int_{n-1}^{2n-1} \frac{1}{x} \, dx.
$$

We have

$$
\int_n^{2n} \frac{1}{x} \, dx = \ln(2n) - \ln n = \ln 2,
$$

$$
\int_{n-1}^{2n-1} \frac{1}{x} \, dx = \ln(2n-1) - \ln(n-1) = \ln \left\{ \frac{2n-1}{n-1} \right\} \to \ln 2.
$$

Hence by the Squeeze Theorem, the desired limit is $\ln 2$.

6. Let $P$ be the limit. Then

$$
\ln(P) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n} \ln \left( 1 + \frac{k}{n} \right)
$$

That sum is a Riemann sum for the following integral:

$$
\int_0^1 \ln(1+x) \, dx = [(1+x)(\ln(1+x) - 1)]_0^1 = 2 \ln 2 - 1.
$$

Hence $P = e^{2\ln 2 - 1} = 4/e$.

7. The series on the left is $xe^{-x^2/2}$. Since the terms of the second sum are non-negative, we can interchange the sum and integral:

$$
\int_0^\infty xe^{-x^2/2} \sum_{n=0}^\infty \frac{x^{2n}}{2^n(n!)^2} \, dx = \sum_{n=0}^\infty \int_0^\infty xe^{-x^2/2} \frac{x^{2n}}{2^n(n!)^2} \, dx
$$
The term for \( n = 0 \) is
\[
\int_0^\infty xe^{-x^2/2} \, dx = \left[ -e^{-x^2/2} \right]_0^\infty = 0 - (-1) = 1.
\]
Next, for \( n \geq 1 \), integrating by parts:
\[
\int_0^\infty x^{2n} \left( xe^{-x^2/2} \right) \, dx = \left[ -x^{2n} e^{-x^2/2} \right]_0^\infty + 2n \int_0^\infty x^{2(n-1)} \left( xe^{-x^2/2} \right) \, dx.
\]
Thus, by induction
\[
\int_0^\infty x^{2n} \left( xe^{-x^2/2} \right) \, dx = 2 \cdot 4 \cdot 6 \cdots 2n.
\]
Hence the integral is
\[
\sum_{n=0}^\infty \frac{1}{2^n n!} = e^{1/2} = \sqrt{e}.
\]

8. The answer is affirmative, in fact any real number \( r \) is the limit of a sequence of numbers of the form \( \sqrt[n]{n} - \sqrt[m]{m} \). First assume
\[
\sqrt[n]{n} \leq r + \sqrt[m]{m} < \sqrt[n]{n} + 1,
\]
which can be accomplished by taking \( n = \lfloor (r + \sqrt[m]{m})^3 \rfloor \). We have
\[
0 \leq r - (\sqrt[n]{n} - \sqrt[m]{m}) < \sqrt[n]{n} + 1 - \sqrt[n]{n} = \frac{1}{\sqrt[(n+1)^2 + \sqrt[(n+1)n + \sqrt[n^2]{n^2}]{n^2}]}.
\]
Since the last expression tends to 0 as \( n \to \infty \), we have that
\[
r = \lim_{m \to \infty} \left\{ \sqrt[(r + \sqrt[m]{m})^3]{n^2} - \sqrt[m]{m} \right\}.
\]

9. From \( f(f(x)) = 1/f(x) \) we get that \( f(y) = 1/y \) for all \( y \in f(\mathbb{R}) \). Hence \( f(999) = 1/999 \). Since \( f \) is continuous it takes all possible values between 1/999 and 999, in particular \( 500 \in f(\mathbb{R}) \). Hence \( f(500) = 1/500 \).

10. Consider the function \( g : [0, 1998/1999] \to \mathbb{R}, g(x) = f(x) - f(x + 1/999) \). Then \( g \) is continuous on \([0, 1998/1999] \), and verifies
\[
\sum_{k=0}^{1998} g(k/1999) = f(1) - f(0) = 0.
\]
Since the sum is zero it is impossible that all its terms are positive or all are negative, so either one is zero, or there are two consecutive terms with opposite signs. In the former case, \( g(k/1999) = 0 \) for some \( k \), so \( f(k/1999) = f((k + 1)/1999) \) and we are done. Otherwise, if there are two consecutive terms \( g(k/1999) \) and \( g((k + 1)/1999) \) with different signs, then for some \( x \in [k/1999, (k + 1)/1999] \) we have \( g(x) = 0 \), hence \( f(x) = f(x + 1/1999) \), and we are also done.
11. The answer is $c \geq 1/2$.

In fact, the given inequality can be written like this:

$$e^{cx^2} - \frac{e^x + e^{-x}}{2} \geq 0.$$ 

The Taylor expansion of the left hand side is

$$e^{cx^2} - \frac{e^x + e^{-x}}{2} = \left( c - \frac{1}{2} \right) x^2 + \left( \frac{c^2}{2!} - \frac{1}{4!} \right) x^4 + \left( \frac{c^3}{3!} - \frac{1}{6!} \right) x^6 + \ldots$$

We see that for $c \geq 1/2$ all the coefficients are non-negative, and the inequality holds.

On the other hand, if $c < 1/2$ we have

$$\lim_{x \to 0} \frac{e^{cx^2} - \frac{e^x + e^{-x}}{2}}{x^2} = c - \frac{1}{2} < 0,$$

so in a neighborhood of 0 the numerator must become negative, and the inequality does not hold.

12. There is no such sequence. If they were convergent their sum would be convergent too, but by the AM-GM inequality we have:

$$\sum_{n=1}^{\infty} \left( a_n + \frac{1}{n^2 a_n} \right) \geq \sum_{n=1}^{\infty} \frac{2}{n} = \infty.$$