EXERCISES

1. Prove that for any positive integer \( n \)
\[
\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1},
\]
where \( \binom{a}{b} = \frac{a!}{b!(a-b)!} \) (binomial coefficient).

2. Prove that for any positive integer \( n \)
\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \binom{2n}{n}.
\]

3. Prove that for any positive integers \( k \leq m, n, \)
\[
\sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j} = \binom{m+n}{k}.
\]

4. Let \( F_n \) be the Fibonacci sequence 0, 1, 2, 3, 5, 8, 13, . . ., defined recursively \( F_0 = 0, \)
\( F_1 = 1, F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2 \). Prove that
\[
\sum_{n=1}^{\infty} \frac{F_n}{2^n} = 2.
\]

5. Find a recurrence for the sequence \( u_n = \) number of nonnegative solutions of
\[
2a + 5b = n.
\]

6. How many different sequences are there that satisfy all the following conditions:
   (a) The items of the sequences are the digits 0–9.
   (b) The length of the sequences is 6 (e.g. 061030)
   (c) Repetitions are allowed.
   (d) The sum of the items is exactly 10 (e.g. 111322).

7. (Leningrad Mathematical Olympiad 1991) A finite sequence \( a_1, a_2, \ldots, a_n \) is called
\( p \)-balanced if any sum of the form \( a_k + a_{k+p} + a_{k+2p} + \cdots \) is the same for any \( k = 1, 2, 3, \ldots, p \). For instance the sequence \( a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 3, a_6 = 2 \)
is 3-balanced because $a_1 + a_4 = 1 + 4 = 5$, $a_2 + a_5 = 2 + 3 = 5$, $a_3 + a_6 = 3 + 2 = 5$.
Prove that if a sequence with 50 members is $p$-balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal zero.
Hints

1. Expand and differentiate \((1 + x)^n\).

2. Expand both sides of \((1 + x)^n(1 + x)^n = (1 + x)^{2n}\) and look at the coefficient of \(x^n\).

3. Expand both sides of \((1 + x)^m(1 + x)^n = (1 + x)^{m+n}\) and look at the coefficient of \(x^j\).

4. Look at the generating function of the Fibonacci sequence.

5. Find the generating function of the sequence \(u_n = \text{number of nonnegative solutions of } 2a + 5b = n\).

6. The answer equals the coefficient of \(x^{10}\) in the expansion of \((1 + x + x^2 + \cdots + x^9)^6\), but that coefficient is very hard to find directly. Try some simplification.

7. Look at the polynomial \(P(x) = a_1 + a_2x + a_3x^2 + \cdots + a_{50}x^{49}\), and at its values at 3rd, 5th, \ldots roots of unity.
Solutions

1. We have

\[ \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = (1+x)^n. \]

Differentiating with respect to \(x\):

\[ \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n} = n(1+x)^{n-1}. \]

Plugging in \(x = 1\) we get the desired identity.

2. The desired expression states the equality between the coefficient of \(x^n\) in each of the following expansions:

\[ (1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k}x^k, \]

and

\[ \{(1+x)^n\}^2 = \left\{ \sum_{k=0}^{n} \binom{n}{k}x^k \right\}^2 = \sum_{k=0}^{n} \sum_{i+j=k} \binom{n}{i} \binom{n}{j}x^k. \]

Taking into account that \(\binom{n}{j} = \binom{n-j}{k}\), for \(k = n\) we get

\[ \sum_{i+j=n} \binom{n}{i} \binom{n}{j} = \sum_{i+j=n} \binom{n}{i} \binom{n-j}{n-i} = \sum_{i=1}^{n} \binom{n}{i}^2, \]

and that must be equal to the coefficient of \(x^n\) in \((1+x)^{2n}\), which is \(\binom{2n}{n}\).

3. This is just a generalization of the previous problem. We have

\[ (1+x)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k}x^k, \]

and

\[ (1+x)^m(1+x)^n = \left\{ \sum_{i=0}^{m} \binom{m}{i}x^i \right\} \left\{ \sum_{j=0}^{n} \binom{n}{j}x^j \right\} \]

\[ = \sum_{k=0}^{m+n} \sum_{i+j=k} \binom{n}{i} \binom{m}{j}x^k. \]

The coefficient of \(x^k\) must be the same on both sides, so:

\[ \binom{m+n}{k} = \sum_{i+j=k} \binom{n}{i} \binom{m}{j} = \sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j}, \]

where we replace \(i = k - j\) in the last step.
4. The generating function for the Fibonacci sequence is
\[ 0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots = \frac{x}{1 - x - x^2}. \]
The desired sum is the left hand side with \( x = 1/2 \), hence its value is
\[ \frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \cdots = \frac{1}{2 - 1/2} = 2. \]

5. The generating function of \( u_n \) is the following:
\[ f(x) = u_0 + u_1 x + u_2 x^2 + \cdots = (1 + x^2 + x^4 + x^6 + \cdots)(1 + x^{10} + x^{15} + \cdots) \]
\[ = \frac{1}{1 - x^2} \frac{1}{1 - x^5} = \frac{1}{1 - x^2 - x^5 + x^7}. \]
Hence
\[ 1 = (1 - x^2 - x^5 + x^7)(u_0 + u_1 x + u_2 x^2 + \cdots). \]
From here we get that \( 1 \cdot u_0 = 1 \), hence \( u_0 = 1 \). Similarly \( 1 \cdot u_1 = 0 \), hence \( u_1 = 0 \), etc., so we get \( u_0 = u_2 = u_4 = u_5 = u_7 = 1 \), and \( u_1 = u_3 = 0 \). Then for \( k > 7 \) the coefficient of \( x^k \) of the product must be
\[ u_k - u_{k-2} - u_{k-5} + u_{k-7} = 0. \]
So we get the following recursive relation for the terms of the sequence:
\[ u_k = u_{k-2} + u_{k-5} - u_{k-7}, \]
together with the initial conditions \( u_0 = u_2 = u_4 = u_5 = u_7 = 1 \), and \( u_1 = u_3 = 0 \).

6. The answer equals the coefficient of \( x^{10} \) in the expansion of
\[ (1 + x + x^2 + \cdots + x^9)^6. \]
Since \( 1 + x + x^2 + \cdots = 1/(1-x) \) the answer can be obtained also from the coefficient of \( x^{10} \) in the Maclaurin series of \( 1/(1-x)^6 = (1-x)^{-6} \). Since that includes six sequences of the form 0, 0, \( \cdots \), 10, \( \cdots \), 0 we need to subtract 6, so the final answer is
\[ \binom{-6}{10} - 6 = \frac{(-6)(-7)(-8)(-9)(-10)(-11)(-12)(-13)(-14)(-15)}{10!} - 6 = 3003 - 6 = 2997. \]

7. Consider the polynomial \( P(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_{50} x^{49} \). If \( r \) is a 3rd root of unity different from 1 then \( P(r) = c(1 + r + r^2) \), where \( c = a_k + a_{k+3} + a_{k+6} + \cdots \). But \( 1 + r + r^2 = (r^3 - 1)/(r - 1) = 0 \), so \( P(r) = 0 \). Analogous reasoning shows that \( P(r) = 0 \) for each 5th, 7th, 11th, 13th, 17th root of unity \( r \) different from 1. Since there are respectively 2 + 4 + 6 + 10 + 12 + 16 = 50 such roots of unity we have that
$P(r)$ is zero for 50 different values of $r$. But a 49-degree polynomial has only 49 roots, so $P(x)$ must be identically zero.