SOME INEQUALITIES

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(Last updated: December 12, 2016)

Introduction. These are a few useful inequalities. Most of them are presented in two versions: in sum form and in integral form. More generally they can be viewed as inequalities involving vectors, the sum version applies to vectors in \( \mathbb{R}^n \) and the integral version applies to spaces of functions.

First a few notations and definitions.

Absolute value. The absolute value of \( x \) is represented \(|x|\).

Norm. Boldface letters line \( \mathbf{u} \) and \( \mathbf{v} \) represent vectors. Their scalar product is represented \( \mathbf{u} \cdot \mathbf{v} \). In \( \mathbb{R}^n \) the scalar product of \( \mathbf{u} = (a_1, \ldots, a_n) \) and \( \mathbf{v} = (b_1, \ldots, b_n) \) is

\[
\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} a_i b_i .
\]

For functions \( f, g : [a, b] \to \mathbb{R} \) their scalar product is

\[
\int_{a}^{b} f(x) g(x) \, dx .
\]

The \( p \)-norm of \( \mathbf{u} \) is represented \( \|\mathbf{u}\|_p \). If \( \mathbf{u} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \), its \( p \)-norm is:

\[
\|\mathbf{u}\|_p = \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} .
\]

For functions \( f : [a, b] \to \mathbb{R} \) the \( p \)-norm is defined:

\[
\|f\|_p = \left( \int_{a}^{b} |f(x)|^p \, dx \right)^{1/p} .
\]

For \( p = 2 \) the norm is called Euclidean.
Convexity. A function \( f : I \to \mathbb{R} \) (\( I \) an interval) is said to be convex if

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]

for every \( x, y \in I \), \( 0 \leq \lambda \leq 1 \). Graphically, the condition is that for \( x < t < y \) the point \((t, f(t))\) should lie below or on the line connecting the points \((x, f(x))\) and \((y, f(y))\).

\[\text{Figure 1. Convex function.}\]

Inequalities

1. **Arithmetic-Geometric Mean Inequality.** (Consequence of convexity of \( e^x \) and Jensen’s inequality.) The geometric mean of positive numbers is not greater than their arithmetic mean, i.e., if \( a_1, a_2, \ldots, a_n > 0 \), then

\[
\left(\prod_{i=1}^{n} a_i\right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} a_i.
\]

Equality happens only for \( a_1 = \cdots = a_n \). (See also the power means inequality.)

2. **Arithmetic-Harmonic Mean Inequality.** The harmonic mean of positive numbers is not greater than their arithmetic mean, i.e., if \( a_1, a_2, \ldots, a_n > 0 \), then

\[
\frac{n}{\sum_{i=1}^{n} 1/a_i} \leq \frac{1}{n} \sum_{i=1}^{n} a_i.
\]

Equality happens only for \( a_1 = \cdots = a_n \).

This is a particular case of the Power Means Inequality.
3. **Cauchy.** (Hölder for \( p = q = 2 \).)

\[ |u \cdot v| \leq \|u\|_2 \|v\|_2. \]

\[ \left| \sum_{i=1}^{n} a_i b_i \right|^2 \leq \left| \sum_{i=1}^{n} a_i^2 \right| \left| \sum_{i=1}^{n} b_i^2 \right|. \]

\[ \left| \int_{a}^{b} f(x)g(x) \, dx \right|^2 \leq \left( \int_{a}^{b} |f(x)|^2 \, dx \right) \left( \int_{a}^{b} |g(x)|^2 \, dx \right). \]

4. **Chebyshev.** Let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be sequences of real numbers which are monotonic in the same direction (we have \( a_1 \leq a_2 \leq \cdots \leq a_n \) and \( b_1 \leq b_2 \leq \cdots \leq b_n \), or we could reverse all inequalities.) Then

\[ \frac{1}{n} \sum_{i=1}^{n} a_i b_i \geq \left( \frac{1}{n} \sum_{i=1}^{n} a_i \right) \left( \frac{1}{n} \sum_{i=1}^{n} b_i \right). \]

Note that \( \text{LHS} - \text{RHS} = \frac{1}{2n^2} \sum_{i,j} (a_i - a_j)(b_i - b_j) \geq 0. \)

5. **Geometric-Harmonic Mean Inequality.** The harmonic mean of positive numbers is not greater than their geometric mean, i.e., if \( a_1, a_2, \ldots, a_n > 0 \), then

\[ \frac{n}{\sum_{i=1}^{n} 1/a_i} \leq \left( \prod_{i=1}^{n} a_i \right)^{1/n}. \]

Equality happens only for \( a_1 = \cdots = a_n \).

This is a particular case of the Power Means Inequality.

6. **Hölder.** If \( p > 1 \) and \( 1/p + 1/q = 1 \) then

\[ |u \cdot v| \leq \|u\|_p \|v\|_q. \]

\[ \left| \sum_{i=1}^{n} a_i b_i \right| \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |b_i|^q \right)^{1/q}. \]

\[ \left| \int_{a}^{b} f(x)g(x) \, dx \right| \leq \left( \int_{a}^{b} |f(x)|^p \, dx \right)^{1/p} \left( \int_{a}^{b} |g(x)|^q \, dx \right)^{1/q}. \]
7. **Jensen.** If \( \varphi \) is convex on \((a, b), x_1, x_2, \ldots, x_n \in (a, b), \lambda_i \geq 0 (i = 1, 2, \ldots, n), \sum_{i=1}^{n} \lambda_i = 1, \) then
\[
\varphi \left( \sum_{i=1}^{n} \lambda_i x_i \right) \leq \sum_{i=1}^{n} \lambda_i \varphi (x_i).
\]

8. **MacLaurin’s Inequalities.** Let \( e_k \) be the \( k \)th degree elementary symmetric polynomial in \( n \) variables:
\[
e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.
\]
Given positive numbers \( a_1, a_2, \ldots, a_n \), let \( S_k = e_k(a_1, a_2, \ldots, a_n) / \binom{n}{k} \) be the averages of the elementary symmetric sums of the \( a_i \). Then
\[
S_1 \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \geq \cdots \geq \sqrt[n]{S_n},
\]
with equality if and only if all the \( a_i \) are equal. (See [3].) (See also Newton’s Inequalities.)

9. **Minkowski.** If \( p > 1 \) then
\[
\| u + v \|_p \leq \| u \|_p + \| v \|_p,
\]
\[
\left( \frac{1}{n} \sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/p} \leq \left( \frac{1}{n} \sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left( \frac{1}{n} \sum_{i=1}^{n} |b_i|^p \right)^{1/p},
\]
\[
\left( \frac{1}{b-a} \int_{a}^{b} |f(x) + g(x)|^p \, dx \right)^{1/p} \leq \left( \frac{1}{b-a} \int_{a}^{b} |f(x)|^p \, dx \right)^{1/p} + \left( \frac{1}{b-a} \int_{a}^{b} |g(x)|^p \, dx \right)^{1/p}.
\]
Equality holds iff \( u \) and \( v \) are proportional.

10. **Muirhead’s Inequality.** Given real numbers \( a_1 \geq \cdots \geq a_n \), and \( b_1 \geq \cdots \geq b_n \), assume that \( \sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \) for \( i = 1, \ldots, n - 1 \), and \( \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \). Then for any nonnegative real numbers \( x_1, \ldots, x_n \), we have
\[
\sum_{\sigma} x_{\sigma_1}^{a_1} \cdots x_{\sigma_n}^{a_n} \leq \sum_{\sigma} x_{\sigma_1}^{b_1} \cdots x_{\sigma_n}^{b_n},
\]
where the sums extend over all permutations \( \sigma \) of \( \{1, \ldots, n\} \) (see theorem 2.18 in [1].)
11. **Newton’s Inequalities.** Let \( e_k \) be the \( k \)th degree elementary symmetric polynomial in \( n \) variables:

\[
e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1}x_{i_2}\cdots x_{i_k}.
\]

Given positive numbers \( a_1, a_2, \ldots, a_n \), let \( S_k = e_k(a_1, a_2, \ldots, a_n)/(n^k) \) be the averages of the elementary symmetric sums of the \( a_i \) for \( k \geq 1 \), and \( S_0 = 1 \). Then (for \( k = 1, 2, \ldots, n - 1 \)):

\[
S_{k-1}S_{k+1} \leq S_k^2,
\]

with equality if and only if all the \( a_i \) are equal. (See also MacLaurin’s Inequalities).

12. **Norm Monotonicity.** If \( a_i > 0 \) (\( i = 1, 2, \ldots, n \)), \( s > t > 0 \), then

\[
\left( \sum_{i=1}^{n} a_i^s \right)^{1/s} \leq \left( \sum_{i=1}^{n} a_i^t \right)^{1/t},
\]

i.e., if \( s > t > 0 \), then \( \|u\|_s \leq \|u\|_t \).

13. **Power Means Inequality.** Let \( r \) be a non-zero real number. We define the \( r \)-mean or \( r \)th power mean of non-negative numbers \( a_1, \ldots, a_n \) as follows:

\[
M^r(a_1, \ldots, a_n) = \left( \frac{1}{n} \sum_{i=1}^{n} a_i^r \right)^{1/r}.
\]

If \( r < 0 \), and \( a_k = 0 \) for some \( k \), we define \( M^r(a_1, \ldots, a_n) = 0 \).

The ordinary arithmetic mean is \( M^1 \), \( M^2 \) is the quadratic mean, \( M^{-1} \) is the harmonic mean. Furthermore we define the 0-mean to be equal to the geometric mean:

\[
M^0(a_1, \ldots, a_n) = \left( \prod_{i=1}^{n} a_i \right)^{1/n}.
\]

Then for any real numbers \( r, s \) such that \( r < s \), the following inequality holds:

\[
M^r(a_1, \ldots, a_n) \leq M^s(a_1, \ldots, a_n).
\]

Equality holds if and only if \( a_1 = \cdots = a_n \), or \( s \leq 0 \) and \( a_k = 0 \) for some \( k \). (See weighted power means inequality).
14. **Power Means Sub/Superadditivity.** We use the definition of $r$-mean given in subsection 13. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be non-negative real numbers.

1. If $r > 1$, then the $r$-mean is **subadditive**, i.e.:
   \[ M^r(a_1 + b_1, \ldots, a_n + b_n) \leq M^r(a_1, \ldots, a_n) + M^r(b_1, \ldots, b_n). \]

2. If $r < 1$, then the $r$-mean is **superadditive**, i.e.:
   \[ M^r(a_1 + b_1, \ldots, a_n + b_n) \geq M^r(a_1, \ldots, a_n) + M^r(b_1, \ldots, b_n). \]

Equality holds if and only if $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are proportional, or $r \leq 0$ and $a_k = b_k = 0$ for some $k$.

15. **Radon’s Inequality.** For real numbers $p > 0$, $x_1, \ldots, x_n \geq 0$, $a_1, \ldots, a_n > 0$, the following inequality holds:
   \[ \sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p}. \]

**Remark:** Radon’s Inequality follows from Hölder’s $|u \cdot v| \leq \|u\|_{p+1}\|v\|_q$, with $u = (x_1/a_1^{1/q}, \ldots, x_n/a_n^{1/q})$, $v = (a_1^{1/q}, \ldots, a_n^{1/q})$, $\frac{1}{p+1} + \frac{1}{q} = 1$.

16. **Rearrangement Inequality.** For every choice of real numbers $x_1 \leq \cdots \leq x_n$ and $y_1 \leq \cdots \leq y_n$, and any permutation $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ of $x_1, \ldots, x_n$, we have
   \[ x_ny_1 + \cdots + x_1y_n \leq x_{\sigma(1)}y_1 + \cdots + x_{\sigma(n)}y_n \leq x_1y_1 + \cdots + x_ny_n. \]

If the numbers are different, e.g., $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, then the lower bound is attained only for the permutation which reverses the order, i.e. $\sigma(i) = n - i + 1$, and the upper bound is attained only for the identity, i.e. $\sigma(i) = i$, for $i = 1, \ldots, n$.

17. **Schur.** If $x, y, z$ are positive real numbers and $k$ is a real number such that $k \geq 1$, then
   \[ x^k(x - y)(x - z) + y^k(y - x)(y - z) + z^k(z - x)(z - y) \geq 0. \]

For $k = 1$ the inequality becomes
   \[ x^3 + y^3 + z^3 + 3xyz \geq xy(x + y) + yz(y + z) + zx(z + x). \]
18. **Schwarz.** (Hölder with $p = q = 2$.)

$$|u \cdot v| \leq \|u\|_2 \|v\|_2,$$

$$\left| \sum_{i=1}^{n} a_i b_i \right|^2 \leq \left( \sum_{i=1}^{n} |a_i|^2 \right) \left( \sum_{i=1}^{n} |b_i|^2 \right),$$

$$\left| \int_{a}^{b} f(x)g(x) \, dx \right|^2 \leq \left( \int_{a}^{b} |f(x)|^2 \, dx \right) \left( \int_{a}^{b} |g(x)|^2 \, dx \right).$$

19. **Strong Mixing Variables Method.** We use the definition of $r$-mean given in subsection 13. Let $F : I \subset \mathbb{R}^n \to \mathbb{R}$ be a symmetric, continuous function satisfying the following: for all $(x_1, x_2, \ldots, x_n) \in I$ such that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$, $F(x_1, x_2, \ldots, x_n) \geq F(t, x_2, \ldots, x_{n-1}, t)$, where $t = M^r(x_1, x_n)$. Then:

$$F(x_1, x_2, \ldots, x_n) \geq F(x, x, \ldots, x),$$

where $x = M^r(x_1, x_2, \ldots, x_n)$.

An analogous result holds replacing $\geq$ with $\leq$.

20. **Weighted Power Means Inequality.** Let $w_1, \ldots, w_n$ be positive real numbers such that $w_1 + \cdots + w_n = 1$. Let $r$ be a non-zero real number. We define the $r$th weighted power mean of non-negative numbers $a_1, \ldots, a_n$ as follows:

$$M^r_w(a_1, \ldots, a_n) = \left( \sum_{i=1}^{n} w_i a_i^r \right)^{1/r}.$$

As $r \to 0$ the $r$th weighted power mean tends to:

$$M^0_w(a_1, \ldots, a_n) = \left( \prod_{i=1}^{n} a_i^{w_i} \right).$$

which we call 0th weighted power mean. If $w_i = 1/n$ we get the ordinary $r$th power means.

Then for any real numbers $r, s$ such that $r < s$, the following inequality holds:

$$M^r_w(a_1, \ldots, a_n) \leq M^s_w(a_1, \ldots, a_n).$$

(If $r, s \neq 0$ note convexity of $x^{s/r}$ and recall Jensen’s inequality.)
1. Various Results

1. Convex Function Superadditivity. If $f : (0, b) \to \mathbb{R}$ is convex, and $f(0) = 0$, then $f$ is superadditive in $[0, b)$, i.e., if $x, y, x+y \in [0, b)$, then $f(x+y) \geq f(x) + f(y)$.

Proof: If $0 \leq \lambda \leq 1$, then by convexity:

$$f(\lambda x) = f(\lambda x + (1-\lambda)0) \leq \lambda f(x) + (1-\lambda)f(0) = \lambda f(x),$$

hence

$$f(x) = f\left(\frac{x}{x+y}(x+y)\right) \leq \frac{x}{x+y}f(x+y)$$

and

$$f(y) = f\left(\frac{y}{x+y}(x+y)\right) \leq \frac{y}{x+y}f(x+y),$$

and adding both inequalities: $f(x) + f(y) \leq f(x+y)$. □

References

