1. Definitions and Notations

1.1. Notations.

*Integer Part, Fractional Part.* The integer part of a real number $x$ is $\lfloor x \rfloor = \text{greatest integer less than or equal to } x$. The fractional part of a real number $x$ is $\langle x \rangle = x - \lfloor x \rfloor$.

1.2. Some Definitions.

*Bernoulli polynomials.* The Bernoulli polynomials $B_n(x)$ can be defined in various ways. The following are two of them:

(1) By a generating function:

$$
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}
$$

(2) By the following recursive formulas ($n \geq 1$):

1. $B_0(x) = 1$,
2. $B'_n(x) = n B_{n-1}(x)$,
3. $\int_0^1 B_n(x) \, dx = 0$.

The first few Bernoulli polynomials are:

- $B_0(x) = 1$
- $B_1(x) = x - \frac{1}{2}$
- $B_2(x) = x^2 - x + \frac{1}{6}$
- $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$

The Bernoulli numbers are $B_n = B_n(0)$. 
2. Various Techniques

2.1. Summation by Parts. This is a formula that resembles integration by parts:

$$\sum_{k=m}^{n} a_k b_k = [a_n S_n - a_m S_{m-1}] + \sum_{k=m}^{n-1} (a_k - a_{k+1}) S_k,$$

where $S_k = \sum_{j=t}^{k} b_j$ (the lower limit is arbitrary).

Example: Prove that the following series converges for every $x$ not an integer:

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i nx}}{n}.$$

Answer: We call

$$S_N = \sum_{n=0}^{N} e^{2\pi i nx} = \frac{e^{2\pi i (N+1)x} - 1}{e^{2\pi i x} - 1} \quad (x \notin \mathbb{Z}),$$

hence

$$|S_N| \leq \frac{2}{|e^{2\pi i x} - 1|}.$$

Next using summation by parts:

$$\sum_{n=1}^{N} \frac{e^{2\pi i nx}}{n} = \frac{1}{N} S_N - \frac{1}{1} S_0 + \sum_{n=1}^{N-1} \left( \frac{1}{n} - \frac{1}{n+1} \right) S_n$$

$$= S_N - 1 + \sum_{n=1}^{N-1} \frac{S_n}{n(n+1)}.$$

Letting $N \rightarrow \infty$ we get a series that converges absolutely by comparison with

$$\sum_{n=1}^{\infty} \frac{2|e^{2\pi i x} - 1|}{n(n+1)} = \frac{2}{|e^{2\pi i x} - 1|} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad \text{(telescopic)} \quad = \frac{2}{|e^{2\pi i x} - 1|}.$$
3. Summation Formulas

3.1. The Euler-Maclaurin Summation Formula. Let \( f : [a, b] \rightarrow \mathbb{C} \) be \( q \) times differentiable. \( \int_{a}^{b} |f^{(q)}(x)| \, dx < \infty \). Then for \( 1 \leq m \leq q \):

\[
\sum_{a < n \leq b} f(n) = \int_{a}^{b} f(x) \, dx + \sum_{k=1}^{m} \frac{(-1)^k}{k!} \left( B_k(\langle b \rangle) f^{(k-1)}(b) - B_k(\langle a \rangle) f^{(k-1)}(a) \right) + \frac{(-1)^{m+1}}{m!} \int_{a}^{b} B_m(\langle x \rangle) f^{(m)}(x) \, dx,
\]

where \( B_k(x) = k \)th Bernoulli polynomial.

Sum of Powers. As an example of application of the Euler-Maclaurin summation formula, we give the sum of the first \( m \)th powers:

\[
S(m, r) = \sum_{n=1}^{m} n^r = 1^r + 2^r + 3^r + \cdots + m^r.
\]

Here \( f(x) = x^r \), so \( f^{(k)}(x) = r!(r-k)/(r-k)! \) for \( k = 0, 1, \ldots, r \), \( f^{(k)}(x) = 0 \) for \( k > r \), and

\[
\sum_{0 < n \leq m} n^r = \int_{0}^{m} x^r \, dx + \sum_{k=1}^{r+1} \frac{(-1)^k}{k!} \left( B_k(\langle m \rangle) f^{(k-1)}(m) - B_k(\langle 0 \rangle) f^{(k-1)}(0) \right)
\]

\[
= \frac{m^{r+1}}{r+1} + \sum_{k=1}^{r+1} \frac{(-1)^k}{k!} B_k \frac{r!}{(r-k+1)!} m^{r-k+1} - \frac{B_{r+1}}{r+1}
\]

\[
= \frac{m^{r+1}}{r+1} + \frac{1}{r+1} \sum_{k=1}^{r+1} \frac{(-1)^k \binom{r+1}{k}}{B_k} m^{r-k+1} - \frac{B_{r+1}}{r+1}
\]

Hence

\[
S(m, r) = \sum_{n=1}^{m} n^r = \frac{1}{r+1} \left\{ \left( \sum_{k=0}^{r+1} \frac{(-1)^k \binom{r+1}{k}}{B_k} m^{r-k+1} \right) - B_{r+1} \right\}
\]

where \( B_k \) are the Bernoulli numbers \( B_0 = 1, B_1 = -1/2, B_3 = 1/6 \), etc.
For instance, for \( r = 2 \) we get:

\[
S(m, 2) = \frac{1}{3} \left\{ \sum_{k=0}^{3} (-1)^{k} \binom{3}{k} B_k m^{3-k} \right\} - B_3
\]

\[
= \frac{1}{3} \left\{ B_0 m^3 - 3B_1 m^2 + 3B_2 m - B_3 - B_3 \right\}
\]

\[
= \frac{1}{3} \left\{ m^3 + \frac{3}{2} m^2 + \frac{1}{2} m \right\}
\]

\[
= \frac{2m^3 + 3m^2 + m}{6}.
\]

3.2. The Poisson Summation Formula. Here \( f \) represents a function \( f : \mathbb{R} \to \mathbb{C} \). The Fourier transform of \( f \) is defined in the following way:

\[
\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t} \, dx.
\]

**Periodic Version of a Function.** The “periodic version” of \( f \) is defined as follows:

\[
f_{\text{per}}(x) = \lim_{N \to \infty} \sum_{n=-N}^{N} f(x + n).
\]

If \( f \) is absolutely integrable over \( \mathbb{R} \), i.e., integrable and \( \int_{-\infty}^{\infty} |f(x)| \, dx < \infty \), then \( f_{\text{per}}(x) \) exists for a.e.\(^1\) \( x \) and is periodic: \( f_{\text{per}}(x + 1) = f_{\text{per}}(x) \).

Furthermore:

\[
\hat{f}(k) = \int_{0}^{1} f_{\text{per}}(x) e^{-2\pi i k x} \, dx.
\]

**Poisson Summation Formula.** If \( f \) is absolutely integrable over \( \mathbb{R} \), of bounded variation and normalized in the sense that for every \( x \),

\[
f(x) = \frac{1}{2} \lim_{h \to 0} \left\{ f(x + h) + f(x - h) \right\},
\]

then

\[
\sum_{n=-\infty}^{\infty} f(x + n) = \lim_{T \to \infty} \sum_{k=-T}^{T} \hat{f}(k) e^{2\pi i k x}.
\]

\(^1\)Every \( x \) but a set of measure zero.