A binary operation on rationals. Let \( \circ \) be a binary operation defined on rational numbers with the following properties:

1. Commutative: \( a \circ b = b \circ a \).
2. Associative: \( a \circ (b \circ c) = (a \circ b) \circ c \).
3. Idempotency of zero: \( 0 \circ 0 = 0 \).
4. Distributivity of ‘+’ respect to ‘\( \circ \)’: \( (a \circ b) + c = (a + c) \circ (b + c) \).

Prove that the operation is either \( a \circ b = \max(a, b) \) or \( a \circ b = \min(a, b) \).

Solution. First we note that every element is idempotent:
\[
a \circ a = (0 + a) \circ (0 + a) = (0 \circ 0) + a = 0 + a = a.
\]

We will say that “\( b \) absorbs \( a \)”, written \( a \preceq b \), if \( a \circ b = b \). This is an order relation, since it is:

1. Reflexive: \( a \circ a = a \Rightarrow a \preceq a \).
2. Antisymmetric: if \( a \preceq b \) and \( b \preceq a \) then \( a \circ b = b \) and \( b \circ c = c \), hence \( a \circ c = a \circ (b \circ c) = (a \circ b) \circ c = b \circ c = c \), so \( a \preceq c \).
3. Transitive: if \( a \preceq b \) and \( b \preceq c \) then \( a \circ b = b \) and \( b \circ c = c \), hence \( a \circ c = a \circ (b \circ c) = (a \circ b) \circ c = b \circ c = c \), so \( a \preceq c \).

Some additional properties of \( \preceq \) are:

1. \( a \preceq b \Rightarrow a + c \preceq b + c \). In fact, because of distributivity of ‘+’ respect to ‘\( \circ \)’ we have:
   \[
a \preceq b \Rightarrow a \circ b = b \Rightarrow (a + c) \circ (b + c) = b + c \Rightarrow a + c \preceq b + c.
   \]
2. \( a \preceq b \) is the least upper bound of \( a \) and \( b \) respect to \( \preceq \), i.e., \( a \preceq a \circ b \) and \( b \preceq a \circ b \), and if \( a \preceq c \) and \( b \preceq c \) then \( a \circ b \preceq c \). In fact:
   \[
a \circ (a \circ b) = (a \circ a) \circ b = a \circ b \Rightarrow a \preceq a \circ b,
   \]
   and similarly for \( b \preceq a \circ b \). On the other hand, if \( a \preceq c \) and \( b \preceq c \) then \( a \circ c = c \) and \( b \circ c = c \), hence \( a \circ b \circ c = a \circ c = c \), so \( a \circ b \preceq c \).

Next, define \( 0^* = \{ x \in \mathbb{Q} \mid 0 \preceq x \} \) (rational numbers absorbing 0). This set has the following properties:

1. It contains zero: \( 0 \in 0^* \).
2. It is closed respect to addition. i.e., \( x, y \in 0^* \Rightarrow x + y \in 0^* \). In fact, if \( x, y \in 0^* \) then \( 0 \preceq x \) and \( 0 \preceq y \), so \( 0 + x \preceq x + y \), i.e., \( x \preceq x + y \), and by transitivity \( 0 \preceq x + y \).
3. It is closed respect to multiplication by positive integers:
   \[
n \in \mathbb{Z}^+ \text{ and } x \in 0^* \Rightarrow n \times x = x + \cdots + x \in 0^*.\]
(4) It is closed respect to division by positive integers: If $x \in 0^*$ and $m \in \mathbb{Z}^+$, consider the $m + 1$ points $x_k = kx/m$ for $k = 0, 1, \ldots, m$. Let $u = x_0 \circ x_1 \circ \cdots \circ x_{m-1}$ and $v = x_1 \circ x_2 \circ \cdots \circ x_m = u + x/m$. We have

$$ u \circ v = x_0 \circ x_1 \circ \cdots \circ x_m, $$

but $x_0 = 0$ and $x_m = x$, and by hypothesis $0 \preceq x$, i.e., $0 \circ x = x$, so

$$ x_0 \circ x_1 \circ \cdots \circ x_m = x_1 \circ \cdots \circ x_m = v, $$

hence $u \circ v = v$, $u \preceq v$, $0 \preceq v - u = x/m$, i.e., $x/m \in 0^*$.

(5) As a consequence of the previous two results, $0^*$ is closed respect to multiplication by positive rational numbers.

(6) It contains some non-zero element: given two different rational numbers $a, b$, we have $a \preceq a \circ b$ and $b \preceq a \circ b$, hence $(a \circ b) - a$ and $(a \circ b) - b$ are in $0^*$, and at least one is not zero.

(7) Either all its non-zero elements are positive, or they are all negative. This can be proved by contradiction: Assume $0^*$ contains both positive and negative elements, i.e., there are $x, y \in 0^*$, $x < 0$, $y > 0$. Since $0^*$ is closed respect to multiplication by positive rational numbers, we have that $x \cdot 1/(-x) = -1$ and $y \cdot 1/y = 1$ are in $0^*$, so $0 \preceq -1$ and $0 \preceq 1$. Adding 1 to the first expression we get $1 \preceq 0$, and by antisymmetry $0 = 1$, which is absurd.

Since $0^*$ is closed respect to multiplication by positive rational numbers there are two possibilities, either $0^*$ consists of all positive rational numbers and zero, or it consists of all negative rational numbers and zero. Assume first that $0^*$ consists of all positive rational numbers and zero. Then $0 \preceq x \iff 0 \preceq x$, or equivalently, $a \preceq b \iff a \leq b$, i.e., the relation $\preceq$ coincides with the usual order on rational numbers. But $a \circ b = \text{least upper bound of } a \text{ and } b$ respect to $\preceq$, which is the same relation as $\leq$, so $a \circ b = \max(a, b)$.

The other possibility is that $0^*$ consists of all negative rational numbers and zero. In this case we have that $a \preceq b \iff a \geq b$, and $a \circ b = \min(a, b)$. □

Remark: The result does not hold on extensions of $\mathbb{Q}$, for instance on $\mathbb{Q}[\sqrt{2}] = \{r + s\sqrt{2} | r, s \in \mathbb{Q}\}$ define the conjugate of an element $x = r + s\sqrt{2}$ as $\overline{x} = r - s\sqrt{2}$. Then the following binary operation verifies the hypotheses of the problem:

$$ x \circ y = \max(\overline{x}, \overline{y}), $$

and is different from $\min(x, y)$ and $\max(x, y)$.

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