**Josephus problem.** A group of \( n \) people are standing in a circle, numbered consecutively clockwise from 1 to \( n \). Starting with person no. 2, we remove every other person, proceeding clockwise. For example, if \( n = 6 \), the people are removed in the order 2, 4, 6, 3, 1, and the last person remaining is no. 5. Let \( j(n) \) denote the last person remaining. Find some simple way to compute \( j(n) \) for any positive integer \( n > 1 \).

**Solution.** Note that the problem does not ask for a ”simple mathematical formula” for \( j(n) \), because the solution is not quite easy to express using only ordinary mathematical symbols, however there is a very simple way to compute \( j(n) \) using binary notation:

\[
j(n) = \text{left rotation of the binary digits of } n
\]

This means that if \( n = x_1x_2x_3 \ldots x_n \), where the \( x_k \) are the digits of the binary representation of \( n \) (with \( x_1 \neq 0 \)) then

\[
j(n) = x_2x_3 \ldots x_nx_1
\]

For instance if \( n = 366 \) (base 10) = 101101110 (in base 2), then we take the '1' in the left and move it to the right: 011011101, so \( j(366) = 011011101 \) (base 2) = 111 (base 10).

For instance for \( n = 6 = 110 \) (base 2) the last person was \( j(6) = 5 = 101 \) (base 2).

The answer also can be expressed as \( j(n) = 2m + 1 \), where \( m = n - 2^k \), \( 2^k \) = maximum power of 2 not exceeding \( n \), i.e., \( 2^k \leq n < 2^{k+1} \).

This can be proved in the following way:

1. First check that if \( n = \text{power of 2} \), say \( n = 2^k \), then the last person remaining is always no. 1. This can be proved by induction on \( k \). For \( k = 1 \) there are only two people, no 2 is removed and no. 1 remains. Then assume that the statement is true for a given \( k \). Assume that \( n = 2^{k+1} \). Then people no. 2, 4, 6, \ldots are removed. After \( 2^k \) removals all even numbered people will have been removed, leaving us with exactly the \( 2^k \) odd numbered people no. 1, 3, 5, \ldots. By induction hypothesis we know that in this case the first person (i.e. no. 1) remains, so the statement (that no. 1 remains if \( n = \text{power of 2} \)) is also true for \( k + 1 \). This completes the induction and proves the statement for every \( k \geq 1 \).

2. Finally, if \( n = 2^k + m \), where \( 0 \leq m < 2^k \leq n \), we start by removing the \( m \) people numbered 2, 4, 6, \ldots, 2m. Now we have a circle with \( 2^k \) people, and the ”first one” (which will remain at the end) at that point is no. 2m + 1.

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\[1\] Equivalently: \( j(n) = 2 \left( n - 2^{\lfloor \log_2 n \rfloor} \right) + 1 \), where \( \lfloor x \rfloor = \text{greatest integer not exceeding } x \).